Steepest Descent Example

Let's solve the following problem with the Steepest Descent Method:

Minimize $f(x_1, x_2, x_3) = (x_1)^2 + x_1(1 - x_2) + (x_2)^2 - x_2x_3 + (x_3)^2 + x_3$

$$\nabla f(\mathbf{x}) = \left[2x_1 + (1 - x_2) \quad -x_1 + 2x_2 - x_3 \quad -x_2 + 2x_3 + 1\right]$$

Let's pick $\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\nabla f(\boldsymbol{x}_0) = [2(0) + (1 - 0), \quad -0 + 2(0) - 0, \quad 0 + 2(0) + 1]$$

$$\nabla f(\boldsymbol{x}_0) = [1, 0, 1]$$

Steepest Descent Example

Let's check to see if the stop criteria is satisfied

Evaluate $||\nabla f(\boldsymbol{x}_0)|| < \varepsilon$

(ε is a small number close to 0, but for this exercise solved manually we can set ε=0.1)

$$\|\nabla f(\boldsymbol{x}_0)\| = \sqrt{1+0+1} = \sqrt{2} > \varepsilon$$

The stop criteria is not satisfied, then find the next point

Steepest Descent Example

Let's solve the following problem with the Steepest Descent Method:

Minimize $f(x_1, x_2, x_3) = (x_1)^2 + x_1(1 - x_2) + (x_2)^2 - x_2x_3 + (x_3)^2 + x_3$

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$$\nabla f(\boldsymbol{x}_0) = [2(0) + (1 - 0), \quad -0 + 2(0) - 0, \quad 0 + 2(0) + 1]$$

$$\boldsymbol{d}_0 = -\nabla f(\boldsymbol{x}_0) = -[1, 0, 1]$$

Steepest Descent Example

$$\boldsymbol{d}_0 = -\nabla f(\boldsymbol{x}_0) = -[1, 0, 1]$$

$$\boldsymbol{x}_1 = [0,0,0] + \alpha_0[-1, 0, -1]$$

$$\boldsymbol{x}_1 = [-\alpha_0, 0, -\alpha_0]$$

Now, we need to determine α_0

Steepest Descent Example

$$\boldsymbol{x}_1 = [-\alpha_0, 0, -\alpha_0]$$

$f(x_1, x_2, x_3) = (x_1)^2 + x_1(1 - x_2) + (x_2)^2 - x_2x_3 + (x_3)^2 + x_3$

$f(-\alpha_0, 0, -\alpha_0) = (-\alpha_0)^2 - \alpha_0(1 - 0) + (0)^2 - 0(-\alpha_0) + (-\alpha_0)^2 + -\alpha_0$

$f(-\alpha_0, 0, -\alpha_0) = 2(\alpha_0)^2 - 2\alpha_0$

$f'(-\alpha_0, 0, -\alpha_0) = 4(\alpha_0) - 2 = 0$ $\qquad \alpha_0 = 1/2$

$$\boldsymbol{x}_1 = \left[-\frac{1}{2}, 0, -\frac{1}{2}\right]$$

Steepest Descent Example

Let's check to see if the stop criteria is satisfied

Evaluate $||\nabla f(\boldsymbol{x}_1)|| < \varepsilon$

(ε is a small number close to 0, but for this exercise solved manually we can set ε=0.1)

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 + (1 - x_2) & -x_1 + 2x_2 - x_3 & -x_2 + 2x_3 + 1 \end{bmatrix}$$

$$\nabla f(\boldsymbol{x}_1) = \left[2\left(-\frac{1}{2}\right) + 1 - 0, +\frac{1}{2} + 2(0) + \frac{1}{2}, 0 + 2(-\frac{1}{2}) + 1\right]$$

$$\nabla f(\boldsymbol{x}_1) = \left[-1 + 1 - 0, +\tfrac{1}{2} + 0 + \tfrac{1}{2}, 0 - 1 + 1\right] = [0, 1, 0]$$

Steepest Descent Example

Let's check to see if the stop criteria is satisfied

Evaluate $||\nabla f(\boldsymbol{x}_1)|| < \varepsilon$

(ε is a small number close to 0, but for this exercise solved manually we can set ε=0.1)

$$\|\nabla f(\boldsymbol{x}_1)\| = \sqrt{0+1+0} = \sqrt{1} > \varepsilon$$

The stop criteria is not satisfied, then find the next point

Steepest Descent Example

Take the negative gradient in $\boldsymbol{x}_1 = \left[-\frac{1}{2}, 0, -\frac{1}{2}\right]$ to find the next search direction:

$$\boldsymbol{d}_1 = -\nabla f(\boldsymbol{x}_1) = -[0, 1, 0]$$

Update the iteration formula: $\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \alpha_k \nabla f(\boldsymbol{x}_k)$

$$\boldsymbol{x}_2 = \left[-\frac{1}{2}, 0, -\frac{1}{2}\right] - \alpha_1 [0, 1, 0]$$

Steepest Descent Example

$$\boldsymbol{x}_2 = \left[-\frac{1}{2}, 0, -\frac{1}{2}\right] + \alpha_1[0, -1, 0]$$

$$\boldsymbol{x}_2 = \left[-\frac{1}{2}, -\alpha_1, -\frac{1}{2}\right]$$

$$f(x_1, x_2, x_3) = (x_1)^2 + x_1(1 - x_2) + (x_2)^2 - x_2x_3 + (x_3)^2 + x_3$$

$$f(-\frac{1}{2}, -\alpha_1, -\frac{1}{2}) = (-\frac{1}{2})^2 - \frac{1}{2}(1+\alpha_1) + (-\alpha_1)^2 \ -\alpha_1\frac{1}{2} + (-\frac{1}{2})^2 - \frac{1}{2}$$

Steepest Descent Example

$$f(-\frac{1}{2}, -\alpha_1, -\frac{1}{2}) = (-\frac{1}{2})^2 - \frac{1}{2}(1+\alpha_1) + (-\alpha_1)^2 \ -\alpha_1 \frac{1}{2} + (-\frac{1}{2})^2 - \frac{1}{2}$$

$$f(-\frac{1}{2}, -\alpha_1, -\frac{1}{2}) = \frac{1}{4} - \frac{1}{2} - \frac{1}{2}\alpha_1 + \alpha_1{}^2 \ -\frac{\alpha_1}{2} + \frac{1}{4} - \frac{1}{2}$$

$$f(-\frac{1}{2}, -\alpha_1, -\frac{1}{2}) = \alpha_1{}^2 - \alpha_1 - \frac{1}{2}$$

$$f'(-\frac{1}{2}, -\alpha_1, -\frac{1}{2}) = 2\alpha_1 - 1 = 0$$

$$\alpha_1 = \frac{1}{2}$$

$$\boldsymbol{x}_2 = \left[-\frac{1}{2}, -\alpha_1, -\frac{1}{2}\right]$$

$$\boldsymbol{x}_2 = \left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right]$$

Steepest Descent Example

Check the stop condition

Evaluate $||\nabla f(\boldsymbol{x}_2)|| < \varepsilon$

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 + (1 - x_2) & -x_1 + 2x_2 - x_3 & -x_2 + 2x_3 + 1 \end{bmatrix}$$

$$\nabla f(\boldsymbol{x}_2) = \left[2\left(-\frac{1}{2}\right) + \left(1 + \frac{1}{2}\right); + \frac{1}{2} + 2\left(-\frac{1}{2}\right) + \frac{1}{2}; + \frac{1}{2} + 2(-\frac{1}{2}) + 1\right]$$

$$\nabla f(\boldsymbol{x}_2) = \left[\frac{1}{2}, \quad 0, \quad \frac{1}{2}\right]$$

Steepest Descent Example

Check the stop condition

Evaluate $||\nabla f(\boldsymbol{x}_2)|| < \varepsilon$

$$\|\nabla f(\boldsymbol{x}_2)\| = \sqrt{\frac{1}{4} + 0 + \frac{1}{4}} > \varepsilon$$

The stop criteria is not satisfied, then find the next point

Steepest Descent Example

Take the negative gradient in $\boldsymbol{x}_2 = \left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right]$ to find the next search direction:

$$\boldsymbol{d}_2 = -\nabla f(\boldsymbol{x}_2) = -\left[\frac{1}{2}, \quad 0, \quad \frac{1}{2}\right]$$

$$\boldsymbol{x}_3 = \left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] - \alpha_2 \left[\frac{1}{2}, \quad 0, \quad \frac{1}{2}\right]$$

$$\boldsymbol{x}_3 = \left[-\frac{1}{2} - \frac{\alpha_2}{2}, -\frac{1}{2}, -\frac{1}{2} + \frac{\alpha_2}{2}\right]$$

Steepest Descent Example

Find α_2 :

$$f(\boldsymbol{x}_3) = \frac{1}{2}(\alpha_2 + 1)^2 - \frac{3}{2}(\alpha_2 + 1) + \frac{1}{4}$$

Set the derivative equal to zero and solve:

$$f'(\boldsymbol{x}_3) = (\alpha_2 + 1)^2 - \frac{3}{2} = 0 \quad \Rightarrow \quad \alpha_2 = \frac{1}{2}$$

$$\boldsymbol{x}_3 = \left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right] - \frac{1}{2}\left[\frac{1}{2}, \quad 0, \quad \frac{1}{2}\right]$$

$$\boldsymbol{x}_3 = \left[-\frac{3}{4}, -\frac{1}{2}, -\frac{3}{4}\right]$$

Steepest Descent Example

Check the stop condition

Evaluate $||\nabla f(\boldsymbol{x}_3)|| < \varepsilon$

$$\nabla f(\boldsymbol{x}_3) = \begin{bmatrix} 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\|\nabla f(\boldsymbol{x}_3)\| = \sqrt{\frac{1}{4}} > \varepsilon$$

The stop criteria is not satisfied, then find the next point

Steepest Descent Example

Take the negative gradient in $\boldsymbol{x}_3$ to find the next search direction:

$$\boldsymbol{d}_3 = -\nabla f(\boldsymbol{x}_3) = -\begin{bmatrix} 0 & \frac{1}{2} & 0 \end{bmatrix}$$

$$\boldsymbol{x}_4 = \left[-\frac{3}{4}, -\frac{1}{2}, -\frac{3}{4}\right] - \alpha_3 \left[0, \quad \frac{1}{2}, \quad 0\right]$$

$$\boldsymbol{x}_4 = \left[-\frac{3}{4}, -\frac{1}{2}(\alpha_3 + 1), -\frac{3}{4}\right]$$

Steepest Descent Example

Find α_3 :

$$f(\boldsymbol{x}_4) = \frac{1}{4}(\alpha_3 + 1)^2 - \frac{3}{2}(\alpha_3) - \frac{3}{2}$$

Set the derivative equal to zero and solve:

$$f'(\boldsymbol{x}_4) = \frac{1}{2}(\alpha_3 + 1) - \frac{9}{8} = 0 \quad \Rightarrow \quad \alpha_3 = \frac{5}{4}$$

$$\boldsymbol{x}_4 = \left[-\frac{3}{4}, -\frac{1}{2}(\alpha_3 + 1), -\frac{3}{4}\right] = \left[-\frac{3}{4}, -\frac{9}{8}, -\frac{3}{4}\right]$$

Steepest Descent Example

Check the stop condition

Evaluate $||\nabla f(\boldsymbol{x}_3)|| < \varepsilon$

$$\nabla f(\boldsymbol{x}_4) = \begin{bmatrix} \frac{5}{8} & -\frac{3}{4} & \frac{5}{8} \end{bmatrix}$$

$$\|\nabla f(\boldsymbol{x}_4)\| > \varepsilon$$

The stop criteria is not satisfied, then find the next point

Steepest Descent Example

Take the negative gradient in $\boldsymbol{x}_4$ to find the next search direction:

$$\boldsymbol{d}_4 = -\nabla f(\boldsymbol{x}_4) = -\begin{bmatrix} \frac{5}{8} & -\frac{3}{4} & \frac{5}{8} \end{bmatrix}$$

$$\boldsymbol{x}_5 = \left[-\frac{3}{4}, -\frac{9}{8}, -\frac{3}{4}\right] - \alpha_4 \left[-\frac{5}{8} \quad \frac{3}{4} \quad -\frac{5}{8}\right]$$

Steepest Descent Example

Find α_4 :

$$f(\boldsymbol{x}_5) = \frac{73}{32}(\alpha_4)^2 - \frac{43}{32}(\alpha_4) - \frac{51}{64}$$

Set the derivative equal to zero and solve:

$$f'(\boldsymbol{x}_5) = \frac{73}{16}\alpha_4 - \frac{43}{32} = 0 \quad \Rightarrow \quad \alpha_4 = \frac{43}{146}$$

$$\boldsymbol{x}_5 = \left[-\frac{3}{4}, -\frac{9}{8}, -\frac{3}{4}\right] + \frac{43}{146}\left[-\frac{5}{8} \quad \frac{3}{4} \quad -\frac{5}{8}\right] = \left[-\frac{1091}{1168} \quad -\frac{66}{73} \quad -\frac{1091}{1168}\right]$$

Steepest Descent Example

Let's check to see if the convergence criteria is satisfied

Evaluate $||\nabla f(\mathbf{x}_5)||$:

$$\nabla f(\mathbf{x}^5) = \begin{bmatrix} \frac{21}{584} & \frac{35}{584} & \frac{21}{584} \end{bmatrix}$$

$$\left\|\nabla f(\mathbf{x}^5)\right\| = \sqrt{\left(21/584\right)^2 + \left(35/584\right)^2 + \left(21/584\right)^2} = 0.0786$$

Steepest Descent Example

So, $||\nabla f(\mathbf{x}^5)|| = 0.0786$, which is very small and we can take it to be close enough to zero for our example

Notice that the answer of

$$\mathbf{x} = \begin{bmatrix} -\frac{1091}{1168} & -\frac{66}{73} & -\frac{1091}{1168} \end{bmatrix}$$

is very close to the value of $\mathbf{x}^* = \begin{bmatrix} -1 & -1 & -1 \end{bmatrix}$

that we can obtain analytically