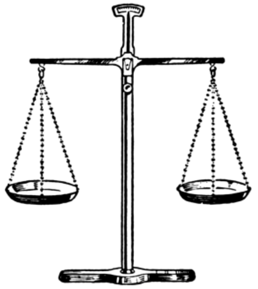


Integer Programming Exercise 1

An Employee Scheduling Problem: Air-Express

Air-Express is an express shipping service that guarantees overnight delivery of packages anywhere in the continental United States.

The company has various operations centers, called hubs, at airports in major cities across the country. Packages are received at hubs from other locations and then shipped to intermediate hubs or to their final destinations.

The manager of the Air-Express hub in Baltimore, Maryland, is concerned about labor costs at the hub and is interested in determining the most effective way to schedule workers.

The hub operates seven days a week, and the number of packages it handles each day varies from one day to the next.

Using historical data on the average number of packages received each day, the manager estimates the number of workers needed to handle the packages as:

An Employee Scheduling Problem: Air-Express

Day of Week	Workers Needed
Sunday	18
Monday	27
Tuesday	22
Wednesday	26
Thursday	25
Friday	21
Saturday	19

An Employee Scheduling Problem: Air-Express

The package handlers working for Air-Express are unionized and are guaranteed a five-day work week with two consecutive days off.

The base wage for the handlers is $655 per week.

Because most workers prefer to have Saturday or Sunday off, the union has negotiated bonuses of $25 per day for its members who work on these days.

These are possible shifts and salaries for package handlers:

Shift	Days Off	Wage
1	Sun & Mon	$680
2	Mon & Tue	$705
3	Tue & Wed	$705
4	Wed & Thr	$705
5	Thr & Fri	$705
6	Fri & Sat	$680
7	Sat & Sun	$655

An Employee Scheduling Problem: Air-Express

The manager wants to keep the total wage expense for the hub as low as possible.

With this in mind, how many package handlers should be assigned to each shift if the manager wants to have a sufficient number of workers available each day?

Defining Decision Variables and Objective Function

X_1 = the number of workers assigned to shift 1

X_2 = the number of workers assigned to shift 2

X_3 = the number of workers assigned to shift 3

X_4 = the number of workers assigned to shift 4

X_5 = the number of workers assigned to shift 5

X_6 = the number of workers assigned to shift 6

X_7 = the number of workers assigned to shift 7

Minimize the total wage expense.

MIN: $680X_1 + 705X_2 + 705X_3 + 705X_4 + 705X_5 + 680X_6 + 655X_7$

Defining the Constraints

- Workers required each day

$$0X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 0X_7 >= 18 \text{ \} Sunday}$$

$$0X_1 + 0X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 >= 27 \text{ \} Monday}$$

$$1X_1 + 0X_2 + 0X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 >= 22 \text{ \}Tuesday}$$

$$1X_1 + 1X_2 + 0X_3 + 0X_4 + 1X_5 + 1X_6 + 1X_7 >= 26 \text{ \} Wednesday}$$

$$1X_1 + 1X_2 + 1X_3 + 0X_4 + 0X_5 + 1X_6 + 1X_7 >= 25 \text{ \} Thursday}$$

$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 0X_5 + 0X_6 + 1X_7 >= 21 \text{ \} Friday}$$

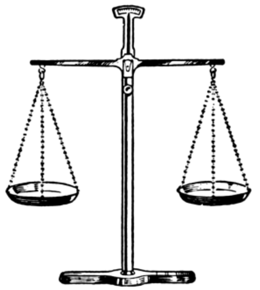
$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 0X_6 + 0X_7 >= 19 \text{ \} Saturday}$$

- Nonnegativity & integrality conditions

$X_i >= 0$ and integer for all i

Binary Variables

- Binary variables are integer variables that can assume only two values: 0 or 1.
- These variables can be useful in a number of practical modeling situations….



Integer Programming
Exercise 2

A Capital Budgeting Problem: CRT Technologies

In a capital budgeting problem, a decision maker is presented with several potential projects or investment alternatives and must determine which projects or investments to choose. The projects or investments typically require different amounts of various resources (for example, money, equipment, personnel) and generate different cash flows to the company. The cash flows for each project or investment are converted to a net present value (NPV). The problem is to determine which set of projects or investments to select to achieve the maximum possible NPV.

Consider the following example:

In his position as vice president of research and development (R&D) for CRT Technologies, Mark Schwartz is responsible for evaluating and choosing which R&D projects to support. The company received 18 R&D proposals from its scientists and engineers, and identified six projects as being consistent with the company's mission.

However, the company does not have the funds available to undertake all six projects. Mark must determine which of the projects to select. The funding requirements for each project are summarized in the following table along with the NPV the company expects each project to generate.

A Capital Budgeting Problem: CRT Technologies

Project	Expected NPV (in $000s)	Capital (in $000s) Required in Year 1	Year 2	Year 3	Year 4	Year 5
1	$141	$75	$25	$20	$15	$10
2	$187	$90	$35	$0	$0	$30
3	$121	$60	$15	$15	$15	$15
4	$83	$30	$20	$10	$5	$5
5	$265	$100	$25	$20	$20	$20
6	$127	$50	$20	$10	$30	$40

- The company has $250,000 available to invest in new projects. It has budgeted $75,000 for continued support for these projects in year 2 and $50,000 per year for years 3, 4, and 5.
- Unused funds in any year cannot be carried over.

A Capital Budgeting Problem: CRT Technologies

The company currently has $250,000 available to invest in new projects.

It has budgeted $75,000 for continued support for these projects in year 2 and $50,000 per year for years 3, 4, and 5.

Surplus funds in any year are reappropriated for other uses within the company and may not be carried over to future years.

Which of the six projects should Mark select ?

Defining the Decision Variables and objective Function

$$X_i = \begin{cases} 1, & \text{if project } i \text{ is selected} \\ 0, & \text{otherwise} \end{cases} \qquad i = 1,2,\ldots,6$$

Maximize the total NPV of selected projects.

$$\text{MAX:} \quad 141X_1 + 187X_2 + 121X_3 + 83X_4 + 265X_5 + 127X_6$$

Defining the Constraints

- Capital Constraints

  $75X_1 + 90X_2 + 60X_3 + 30X_4 + 100X_5 + 50X_6 <= 250$ } year 1

  $25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 <= 75$ } year 2

  $20X_1 + 0X_2 + 15X_3 + 10X_4 + 20X_5 + 10X_6 <= 50$ } year 3

  $15X_1 + 0X_2 + 15X_3 + 5X_4 + 20X_5 + 30X_6 <= 50$ } year 4

  $10X_1 + 30X_2 + 15X_3 + 5X_4 + 20X_5 + 40X_6 <= 50$ } year 5

- Binary Constraints

  All X_i must be binary

A Capital Budgeting Problem: CRT Technologies

An approach to solving this problem is to create a ranked list of the projects in decreasing order by NPV and then select projects from this list, in order, until the capital is depleted.

If we apply this heuristic to the current problem, we would select projects 5 and 2, but we could not select any more projects due to a lack of capital in year 5.

This solution would generate a total NPV of $452,000.

Is this the optimal solution ?

Binary Variables & Logical Conditions

- Binary variables are also useful in modeling a number of logical conditions.
- How can we formulate the following conditions ?

  - Of projects 1, 3 & 6, no more than one may be selected
  - Of projects 1, 3 & 6, exactly one must be selected
  - Project 4 cannot be selected unless project 5 is also selected

Binary Variables & Logical Conditions

- Binary variables are also useful in modeling a number of logical conditions.
- How can we formulate the following conditions ?

  - Of projects 1, 3 & 6, no more than one may be selected
    $X_1 + X_3 + X_6 <= 1$
  - Of projects 1, 3 & 6, exactly one must be selected
    $X_1 + X_3 + X_6 = 1$
  - Project 4 cannot be selected unless project 5 is also selected
    $X_4 – X_5 <= 0$

Integer Programming
Exercise 3

The Fixed-Charge Problem

- Many decisions result in a fixed or lump-sum cost being incurred:
  - The cost to lease, rent, or purchase a piece of equipment or a vehicle that will be required if a particular action is taken.
  - The setup cost required to prepare a machine or to produce a different type of product.
  - The cost to construct a new production line that will be required if a particular decision is made.
  - The cost of hiring additional personnel that will be required if a particular decision is made.

Example Fixed-Charge Problem: Remington Manufacturing

Remington Manufacturing is planning its next production cycle.
The company can produce three products, each of which must undergo machining, grinding, and assembly operations.
The following table below summarizes the hours of machining, grinding, and assembly required by each unit of each product, and the total hours of capacity available for each operation.

	Hours Required By:			
Operation	Prod. 1	Prod. 2	Prod. 3	Hours Available
Machining	2	3	6	600
Grinding	6	3	4	300
Assembly	5	6	2	400
Unit Profit	$48	$55	$50	
Setup Cost	$1000	$800	$900	

Example Fixed-Charge Problem: Remington Manufacturing

The cost accounting department has estimated that each unit of product 1 manufactured and sold will contribute \$48 to profit, and each unit of products 2 and 3 contributes \$55 and \$50, respectively.

However, manufacturing a unit of product 1 requires a setup operation on the production line that costs \$1,000.

Similar setups are required for products 2 and 3 at costs of \$800 and \$900, respectively.

The marketing department believes that it can sell all the products produced. Therefore, the management of Remington wants to determine the most profitable mix of products to produce.

Formulate the problem.

Defining Decision Variables and Objective Function

X_i = the amount of product i to be produced, i = 1, 2, 3

$$Y_i = \begin{cases} 1, \text{ if } X_i > 0 \\ 0, \text{ if } X_i = 0 \end{cases} \qquad i = 1, 2, 3$$

Maximize total profit.

$$\text{MAX:} \quad 48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$$

Defining the Constraints

- Resource Constraints

  $2X_1 + 3X_2 + 6X_3 <= 600$ } machining

  $6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

  $5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

- Binary Constraints

  All Y_i must be binary

- Nonnegativity conditions

  $X_i >= 0,\ i = 1, 2, ..., 6$

- Is there a missing link?

Defining the Constraints *(cont'd)*

- Linking Constraints (with "Big M")

  $X_1 <= M_1Y_1$ or $X_1 - M_1Y_1 <= 0$

  $X_2 <= M_2Y_2$ or $X_2 - M_2Y_2 <= 0$

  $X_3 <= M_3Y_3$ or $X_3 - M_3Y_3 <= 0$
- If $X_i > 0$ these constraints force the associated Y_i to equal 1.
- If $X_i = 0$ these constraints allow Y_i to equal 0 or 1, but the objective will cause Solver to choose 0.
- Note that M_i imposes an upper bounds on X_i.
- It helps to find reasonable values for the M_i.

Finding Reasonable Values for M_1

- Consider the resource constraints

$2X_1 + 3X_2 + 6X_3 <= 600$ } machining

$6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

$5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

- What is the maximum value X_1 can assume?

Let $X_2 = X_3 = 0$

$X_1 = \text{MIN}(600/2, 300/6, 400/5)$

$= \text{MIN}(300, 50, 80)$

$= 50$

- Maximum values for X_2 & X_3 can be found similarly.

Summary of the Model

MAX: $48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$

S.T.:

$2X_1 + 3X_2 + 6X_3 <= 600$ } machining

$6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

$5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

$$\left.\begin{aligned} X_1 - 50Y_1 &<= 0 \\ X_2 - 67Y_2 &<= 0 \\ X_3 - 75Y_3 &<= 0 \end{aligned}\right\} \text{linking constraints}$$

All Y_i must be binary

$X_i >= 0, i = 1, 2, 3$

Minimum Order Size Restrictions

Suppose Remington doesn't want to manufacture any units of product 3 unless it produces at least 40 units...

How can you model this requirement ?

Minimum Order Size Restrictions

Suppose Remington doesn't want to manufacture any units of product 3 unless it produces at least 40 units...

How can you model this requirement ?

$$X_3 <= M_3 Y_3$$

$$X_3 >= 40\ Y_3$$

Integer Programming Exercise 4

Quantity Discounts

- Assume…

  If Blue Ridge Hot Tubs produces more than 75 Aqua-Spas, it obtains discounts that increase the unit profit to $375.

  If it produces more than 50 Hydro-Luxes, the profit increases to $325.

Quantity Discount Model

MAX: $350X_{11} + 375X_{12} + 300X_{21} + 325X_{22}$

S.T.: $1X_{11} + 1X_{12} + 1X_{21} + 1X_{22} <= 200$ } pumps

$9X_{11} + 9X_{12} + 6X_{21} + 6X_{22} <= 1566$ } labor

$12X_{11} + 12X_{12} + 16X_{21} + 16X_{22} <= 2880$ } tubing

$X_{12} <= M_{12}Y_1$

$X_{11} >= 75Y_1$

$X_{22} <= M_{22}Y_2$

$X_{21} >= 50Y_2$

$X_{ij} >= 0$

X_{ij} must be integers, Y_i must be binary

Integer Programming Exercise 5

A Contract Award Problem

- B&G Construction has 4 building projects and can purchase cement from 3 companies for the following costs:

	Cost per Delivered Ton of Cement				Max.
	Project 1	Project 2	Project 3	Project 4	Supply
Co. 1	\$120	\$115	\$130	\$125	525
Co. 2	\$100	\$150	\$110	\$105	450
Co. 3	\$140	\$95	\$145	\$165	550
Needs (tons)	450	275	300	350	

A Contract Award Problem

In addition to the maximum supplies listed, each cement company placed special conditions on its bid.

Specifically:

Company 1 will not supply orders of less than 150 tons for any project

Company 2 can supply more than 200 tons to no more than one of the projects

Company 3 will accept only orders that total 200, 400, or 550 tons

B&G can contract with more than one supplier to meet the cement requirements for a given project.

The problem is to determine what amounts to purchase from each supplier to meet the demands for each project at the least total cost.

Defining Decision Variables and Obj. Function

X_{ij} = tons of cement purchased from company i for project j

Minimize total cost

$$\begin{aligned}\text{MIN:}\quad & 120X_{11} + 115X_{12} + 130X_{13} + 125X_{14} \\ & + 100X_{21} + 150X_{22} + 110X_{23} + 105X_{24} \\ & + 140X_{31} + 95X_{32} + 145X_{33} + 165X_{34}\end{aligned}$$

Defining the Constraints

- Supply Constraints

$$X_{11} + X_{12} + X_{13} + X_{14} <= 525 \quad \} \text{ company 1}$$
$$X_{21} + X_{22} + X_{23} + X_{24} <= 450 \quad \} \text{ company 2}$$
$$X_{31} + X_{32} + X_{33} + X_{34} <= 550 \quad \} \text{ company 3}$$

- Demand Constraints

$$X_{11} + X_{21} + X_{31} = 450 \quad \} \text{ project 1}$$
$$X_{12} + X_{22} + X_{32} = 275 \quad \} \text{ project 2}$$
$$X_{13} + X_{23} + X_{33} = 300 \quad \} \text{ project 3}$$
$$X_{14} + X_{24} + X_{34} = 350 \quad \} \text{ project 4}$$

Defining the Constraints-I

- Company 1 Side Constraints

$X_{11} <= 525Y_{11}$

$X_{12} <= 525Y_{12}$

$X_{13} <= 525Y_{13}$

$X_{14} <= 525Y_{14}$

$X_{11} >= 150Y_{11}$

$X_{12} >= 150Y_{12}$

$X_{13} >= 150Y_{13}$

$X_{14} >= 150Y_{14}$

Y_{ij} binary

Defining the Constraints-II

- Company 2 Side Constraints

  $X_{21} <= 200 + 250Y_{21}$

  $X_{22} <= 200 + 250Y_{22}$

  $X_{23} <= 200 + 250Y_{23}$

  $X_{24} <= 200 + 250Y_{24}$

  $Y_{21} + Y_{22} + Y_{23} + Y_{24} <= 1$

  Y_{ij} binary

- Company 3 Side Constraints

  $X_{31} + X_{32} + X_{33} + X_{34} = 200Y_{31} + 400Y_{32} + 550Y_{33}$

  $Y_{31} + Y_{32} + Y_{33} <= 1$