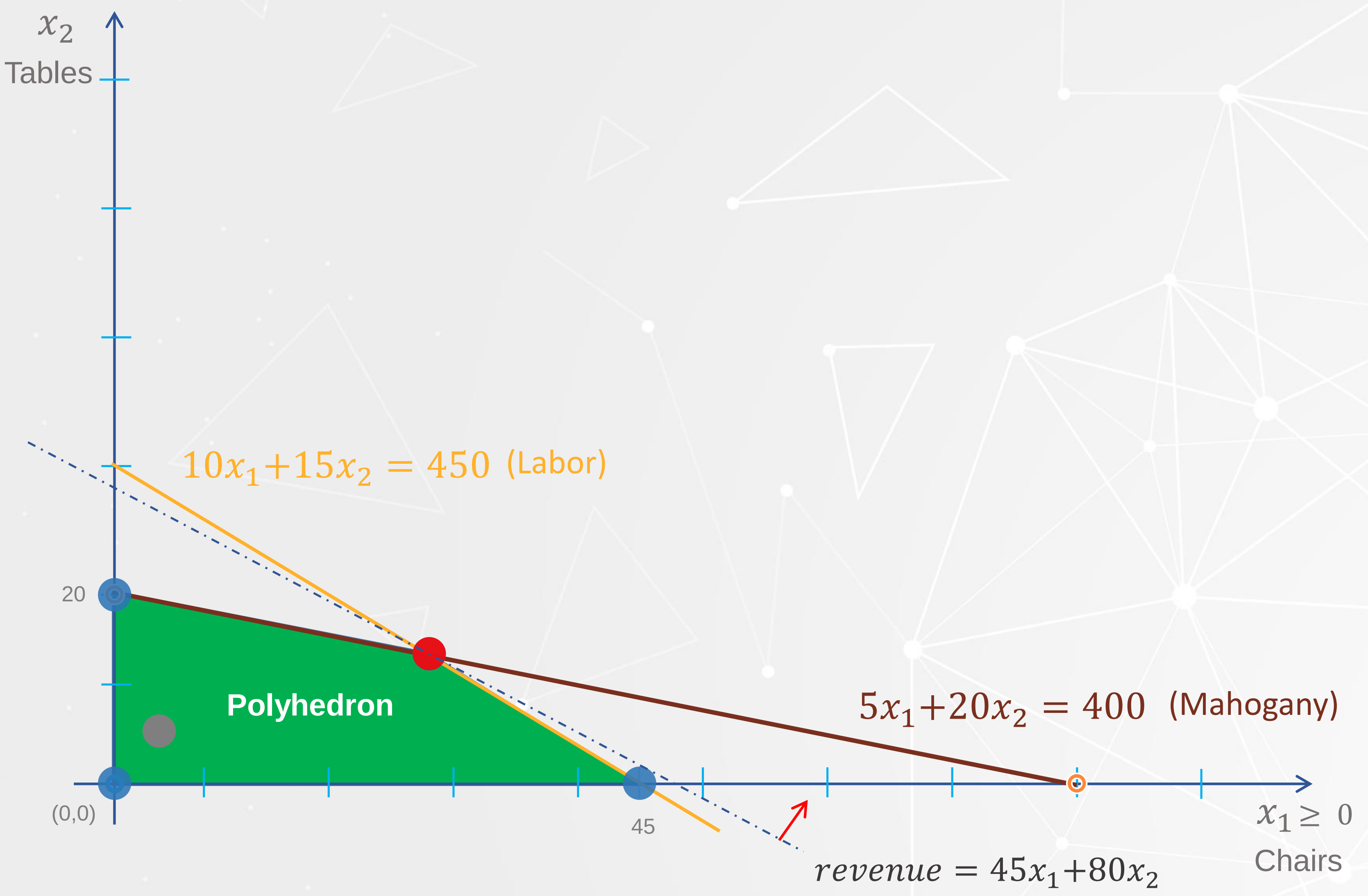
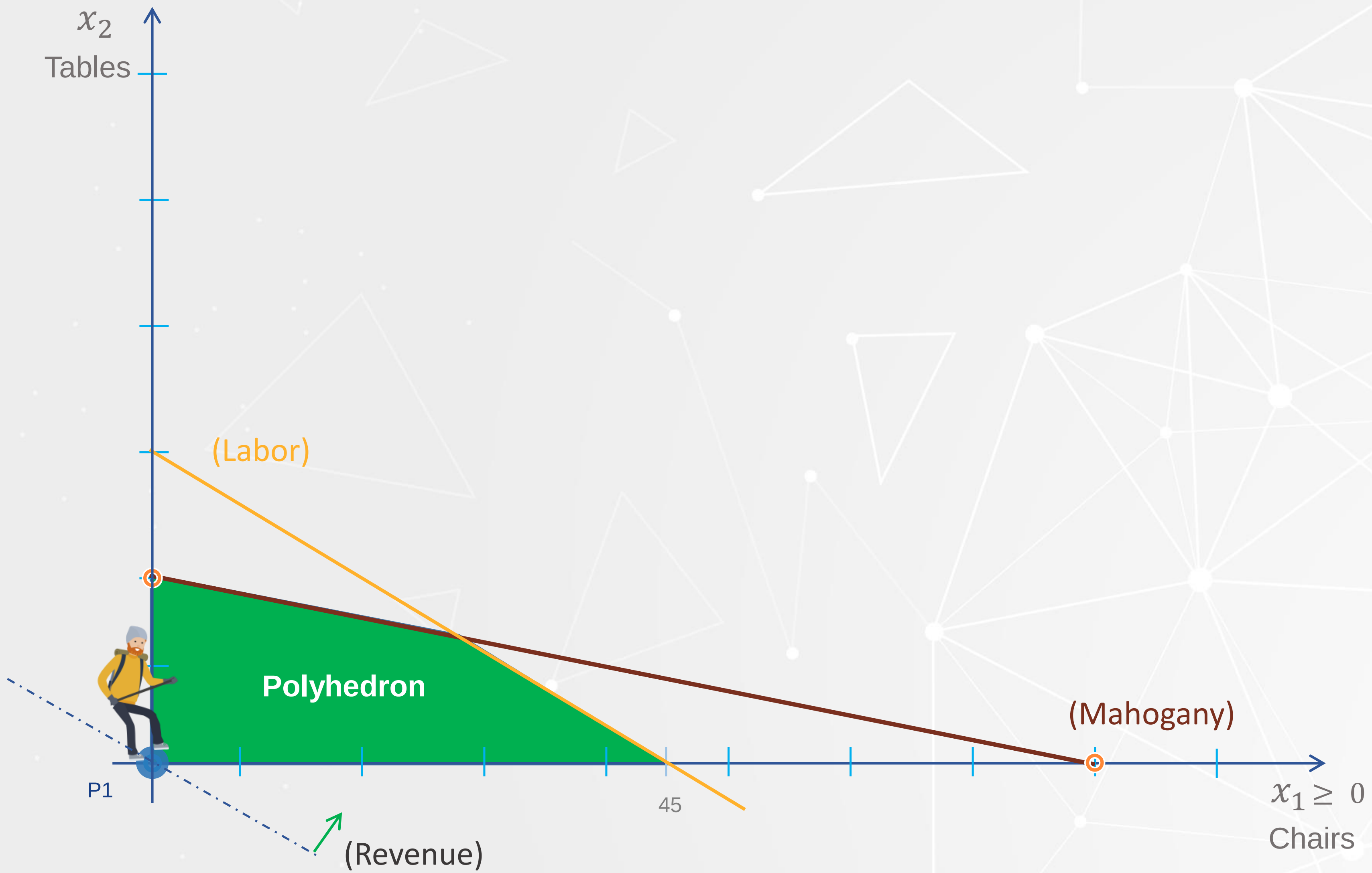
Fundamental theorem of linear programming ..



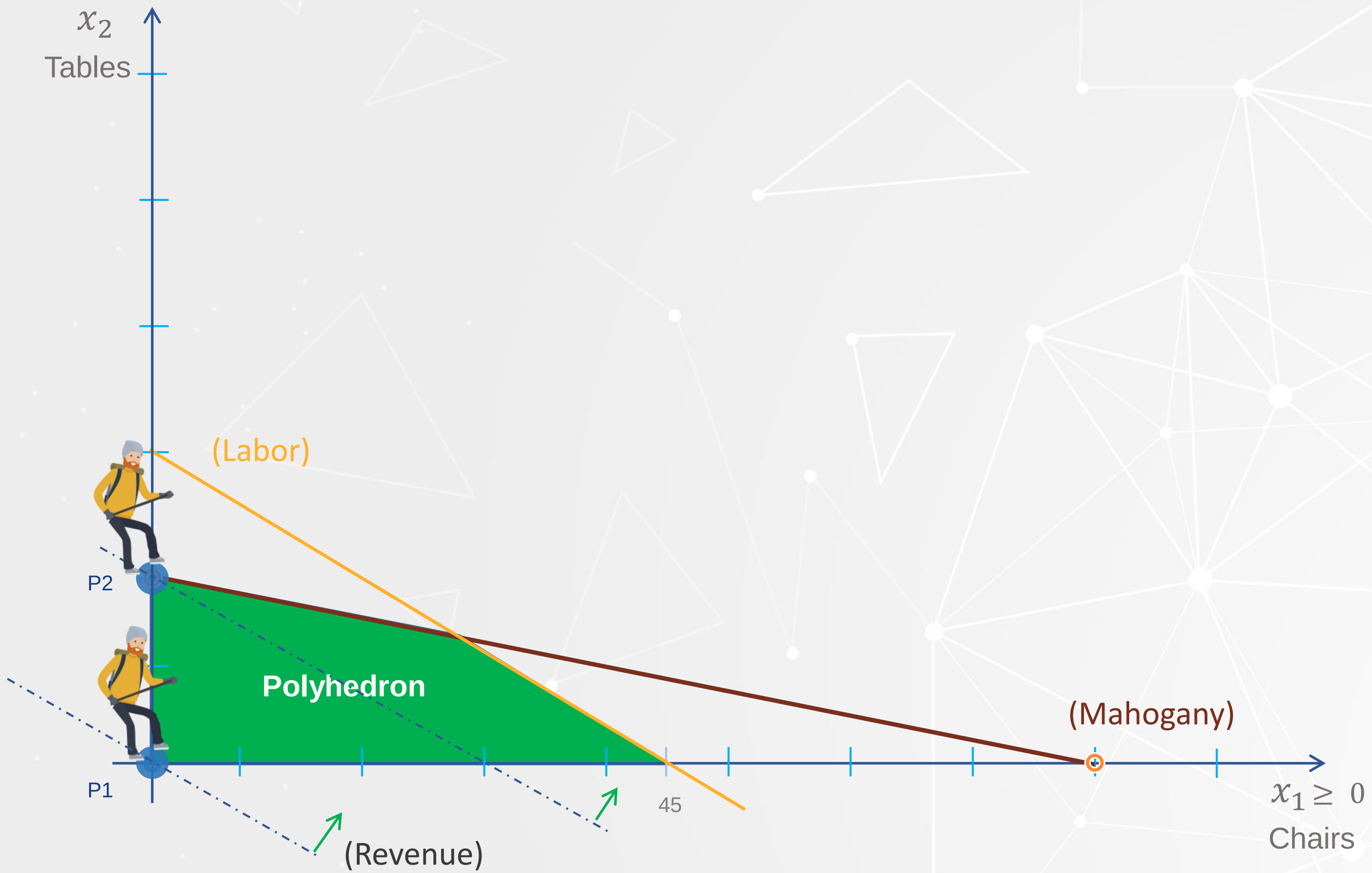
- **Definitions**:
  - A **solution** of an LP problem is a set of values of the decision variables that satisfies all the constraints of the problem defined by the polyhedron.
  - A **corner point solution** is a vertex of the polyhedron defining the feasible region of the LP problem.
  - An **optimal solution** is a solution of the LP problem that cannot be improved.
- **Theorem**:
  - If a linear programming problem has an optimal solution, there is at least one optimal solution that is a corner point solution.

Fundamental theorem of linear programming .. 2



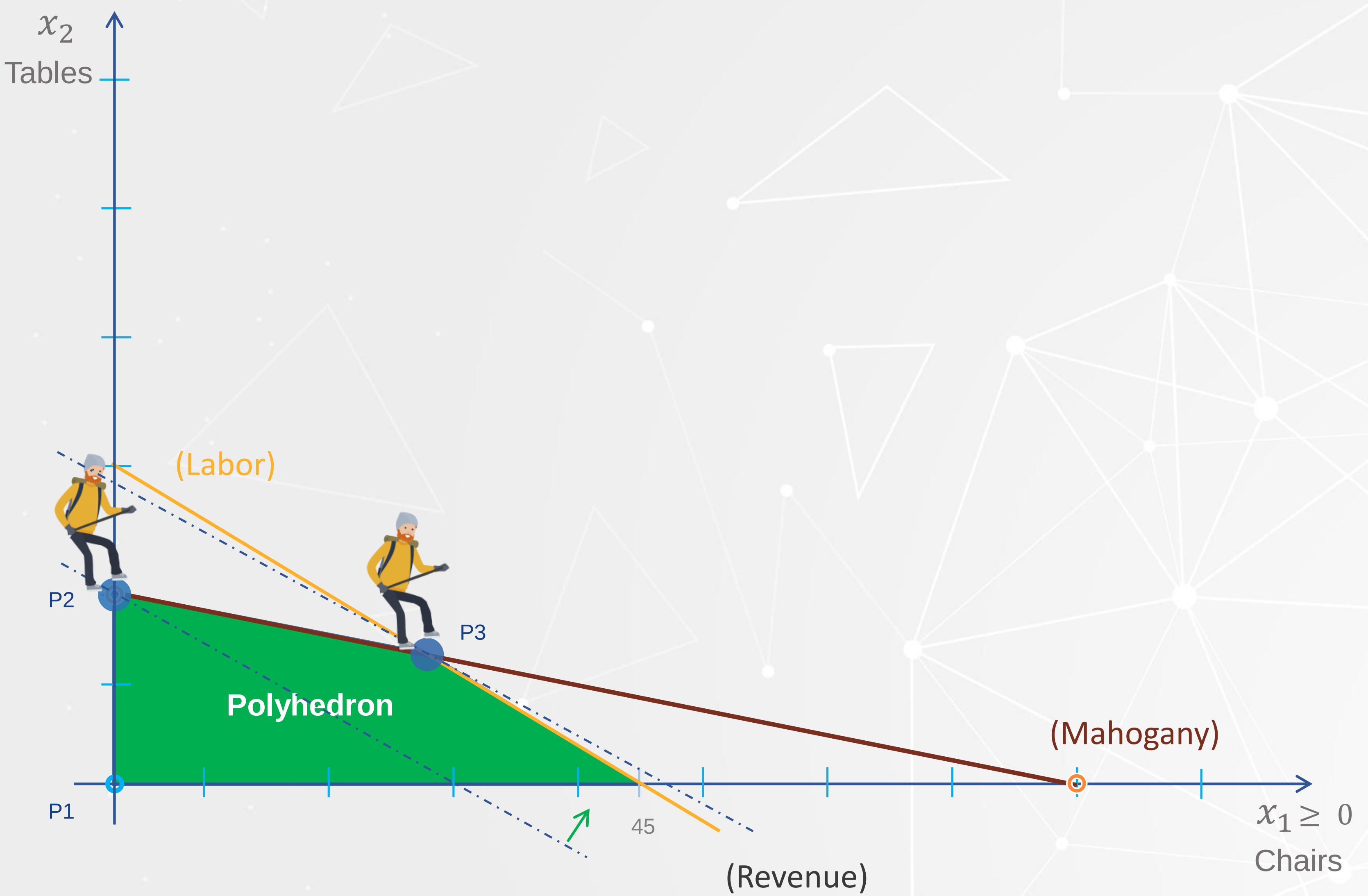
- **Theorem**:
  - If a linear programming problem has an optimal solution, there is at least one optimal solution that is a corner point solution.
- Initial corner point solution P1 =(0 chairs, 0 tables)

Fundamental theorem of linear programming .. 2



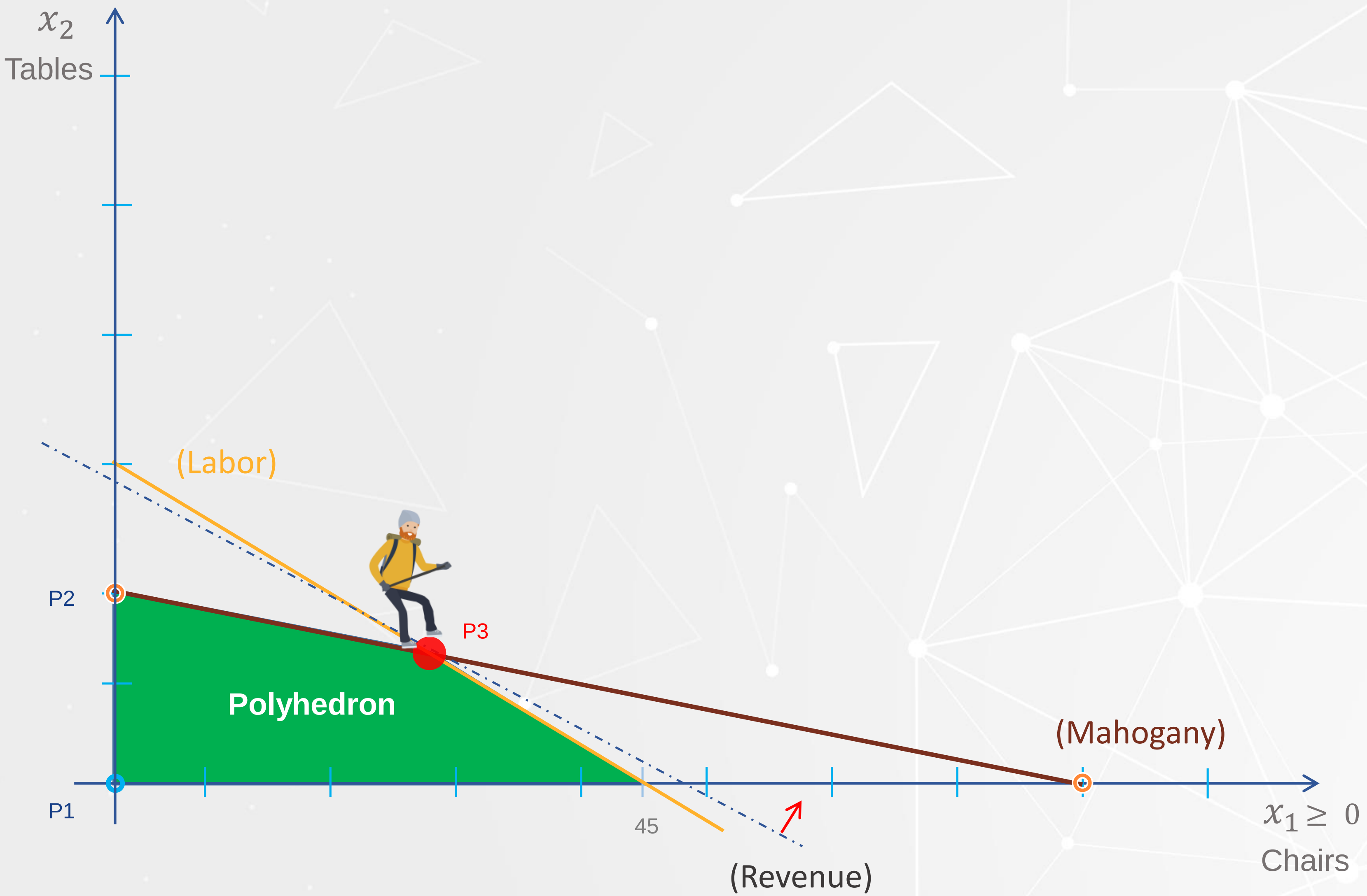
- **Theorem**:
  - If a linear programming problem has an optimal solution, there is at least one optimal solution that is a corner point solution.
- Initial corner point solution
  P1 =(0 chairs, 0 tables)
- Adjacent corner point solution
  P2 =(0 chairs, 20 tables)

Fundamental theorem of linear programming .. 2



- **Theorem**:
  - If a linear programming problem has an optimal solution, there is at least one optimal solution that is a corner point solution.
- Initial corner point solution
  P1 =(0 chairs, 0 tables)
- Adjacent corner point solution
  P2 =(0 chairs, 20 tables)
- Adjacent corner point solution
  P3 =(24 chairs, 14 tables)

Fundamental theorem of linear programming .. 2



- **Theorem**:
  - If a linear programming problem has an optimal solution, there is at least one optimal solution that is a corner point solution.
- Initial corner point solution
  P1 =(0 chairs, 0 tables)
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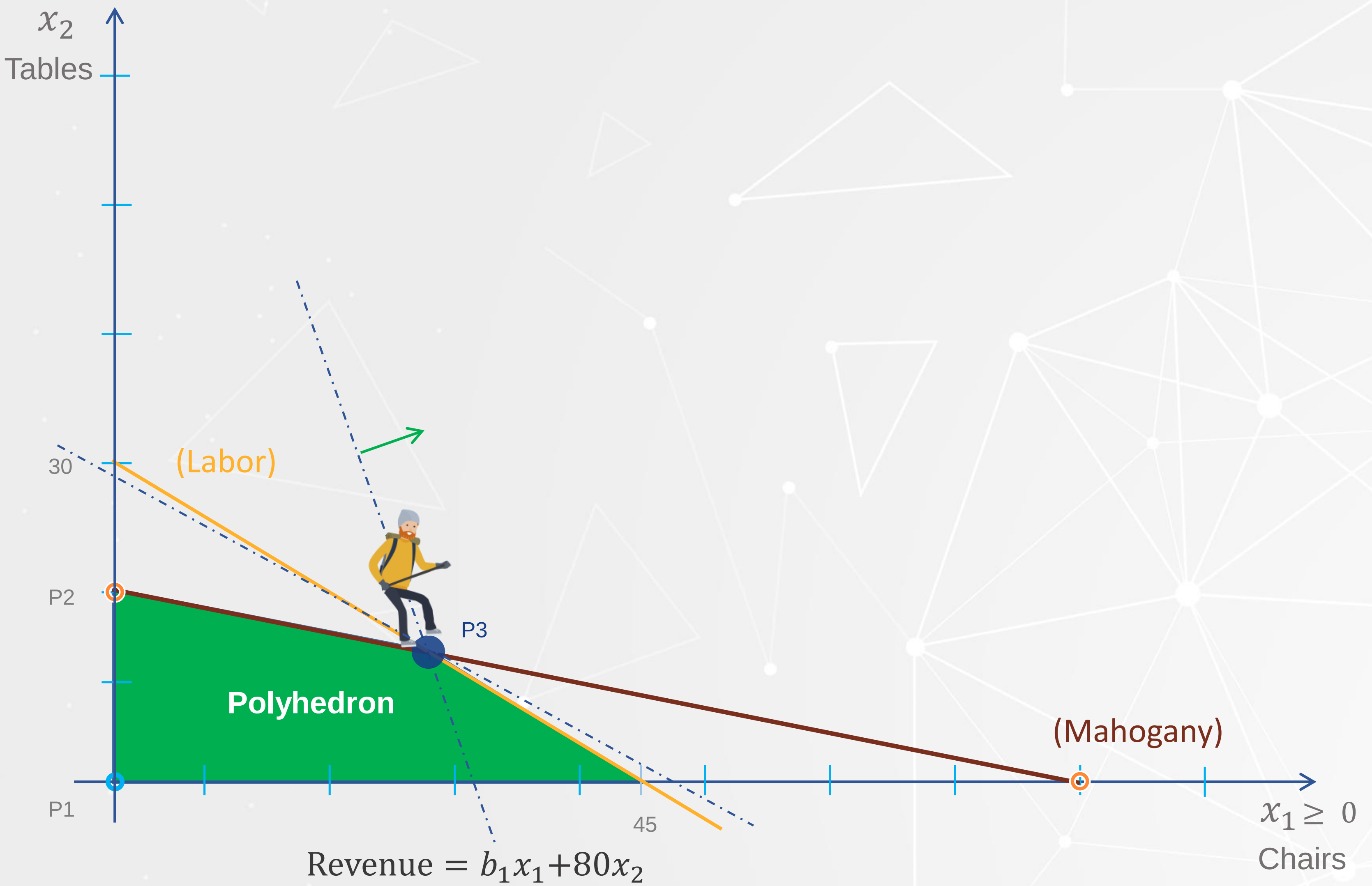
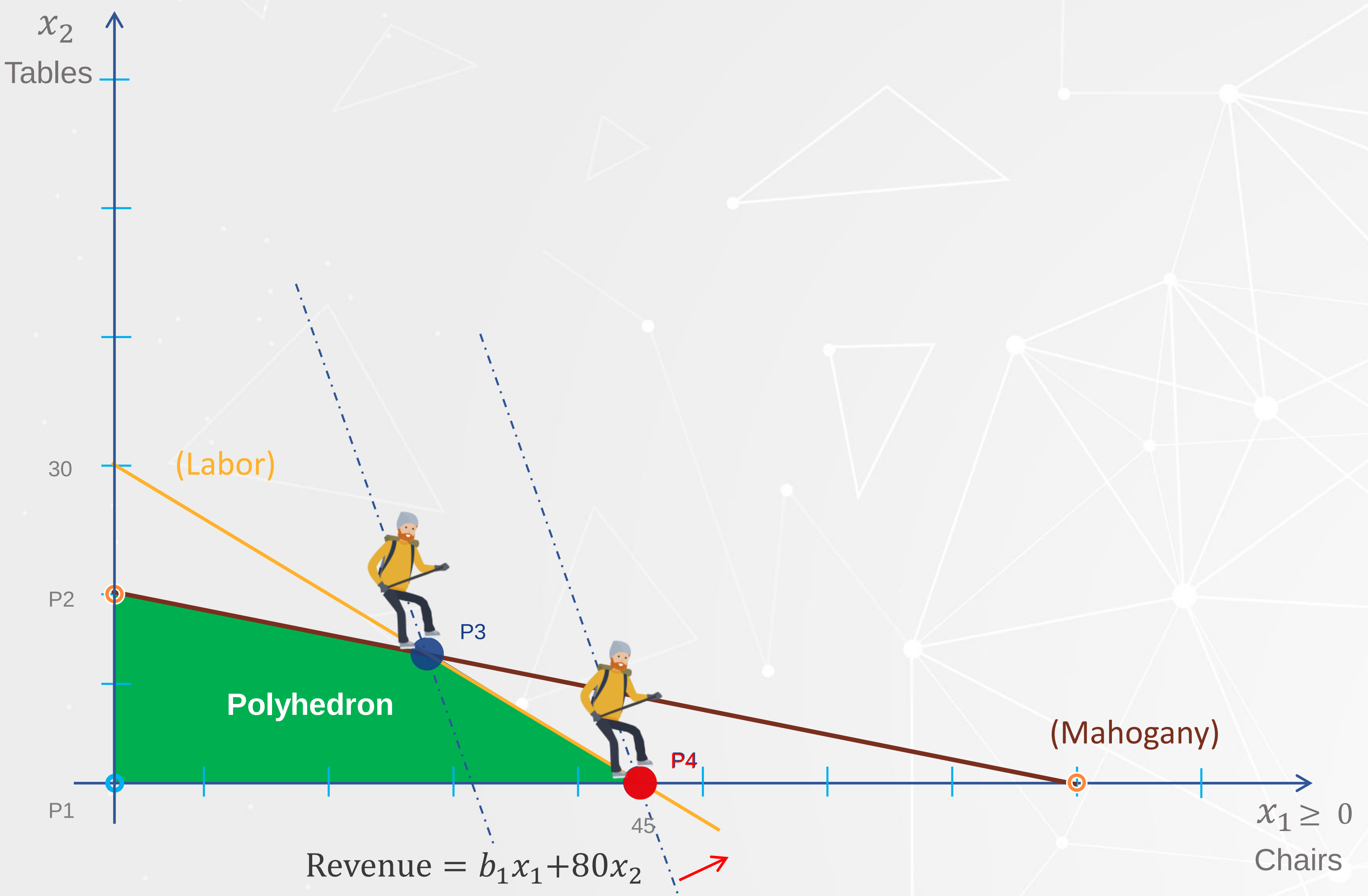
  P2 =(0 chairs, 20 tables)
- Adjacent corner point solution
  P3 =(24 chairs, 14 tables)

Fundamental theorem of linear programming .. 3



- Significant increase in the price (b1) of chairs.
- New oportunity to increase revenue

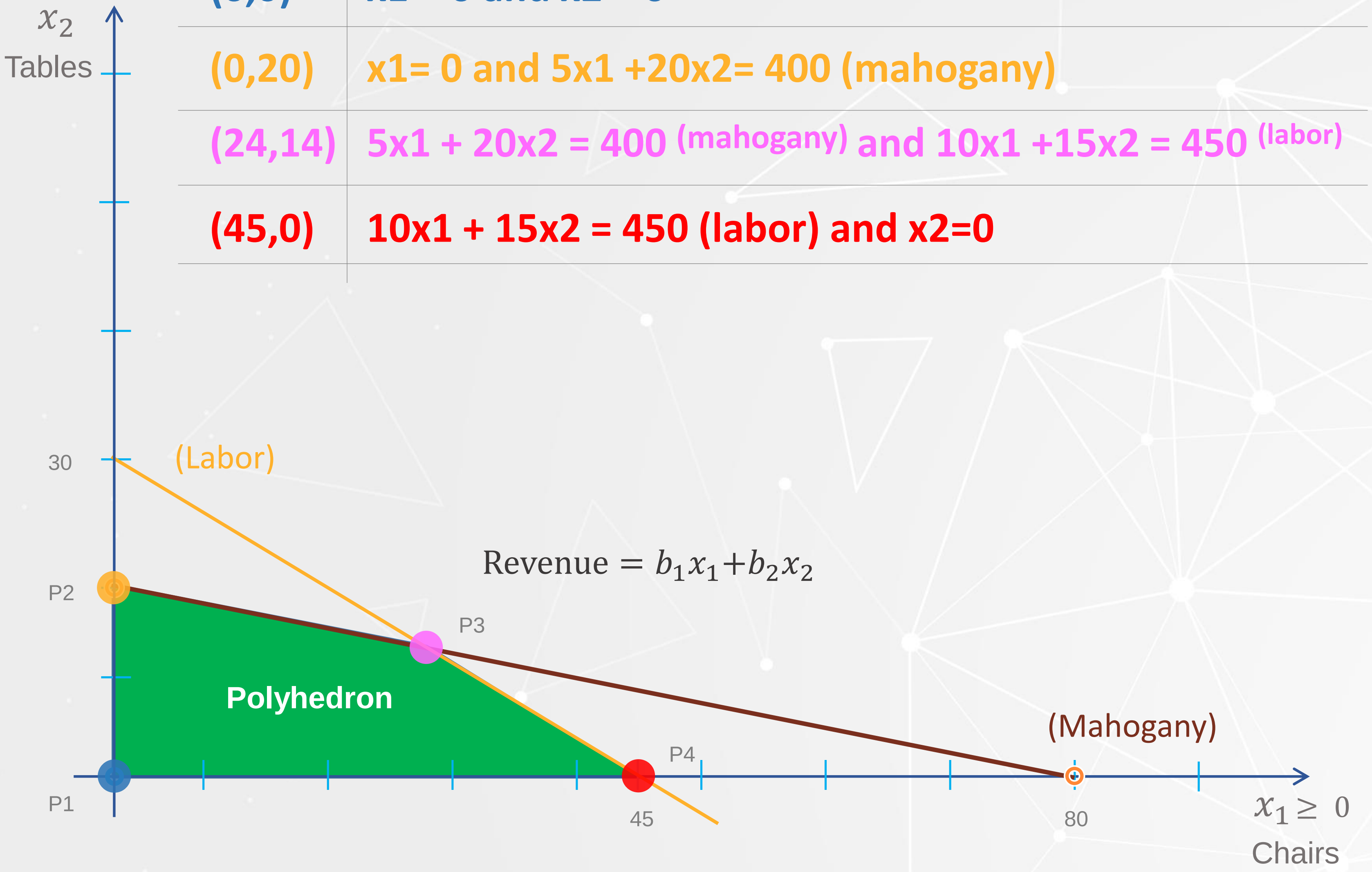
Fundamental theorem of linear programming .. 3



- The new Production Plan P4 = (45 chairs, 0 tables) is optimal

Fundamental theorem of linear programming .. 4

Vertex	Equations
(0,0)	x1 = 0 and x2 = 0
(0,20)	x1= 0 and 5x1 +20x2= 400 (mahogany)
(24,14)	5x1 + 20x2 = 400 (mahogany) and 10x1 +15x2 = 450 (labor)
(45,0)	10x1 + 15x2 = 450 (labor) and x2=0



- **Theorem**:
  - If a linear programming problem has an optimal solution, there is at least one optimal solution that is a corner point solution.

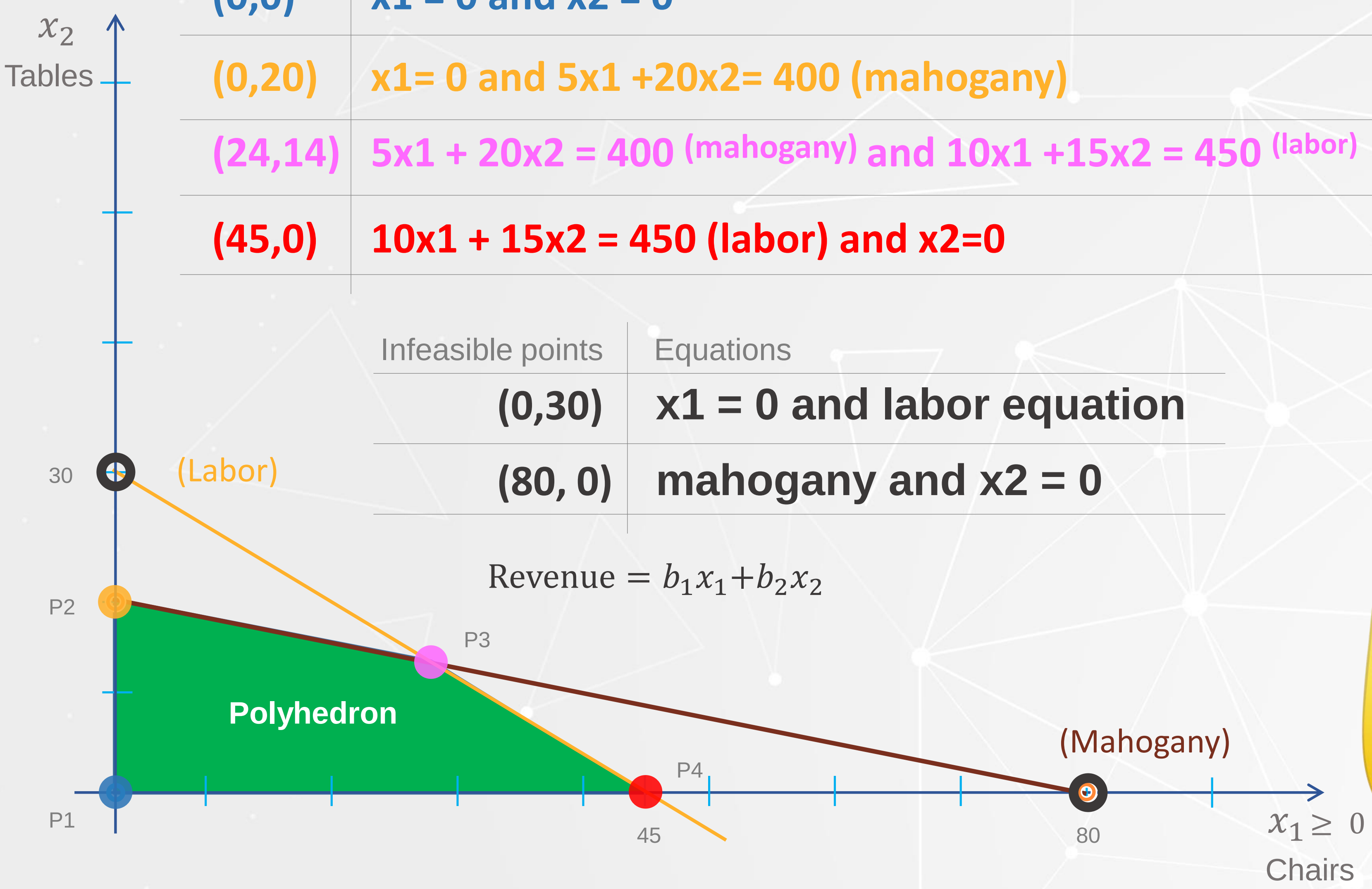
Note that the vertices (corner points) of the polyhedron are the solution of a system of equations.

Fundamental theorem of linear programming .. 5

Vertex	Equations
(0,0)	x1 = 0 and x2 = 0
(0,20)	x1= 0 and 5x1 +20x2= 400 (mahogany)
(24,14)	5x1 + 20x2 = 400 (mahogany) and 10x1 +15x2 = 450 (labor)
(45,0)	10x1 + 15x2 = 450 (labor) and x2=0

Infeasible points	Equations
(0,30)	x1 = 0 and labor equation
(80, 0)	mahogany and x2 = 0

$$\text{Revenue} = b_1 x_1 + b_2 x_2$$



- **Theorem**:
  - If a linear programming problem has an optimal solution, there is at least one optimal solution that is a corner point solution.

Note that the vertices (corner points) of the polyhedron are the solution of a system of equations.

Also, observe that there are other points that are the solutions of a system of equations, although these points are infeasible because they are not vertices of the polyhedron.

GUROBI OPTIMIZATION

Enumeration approach

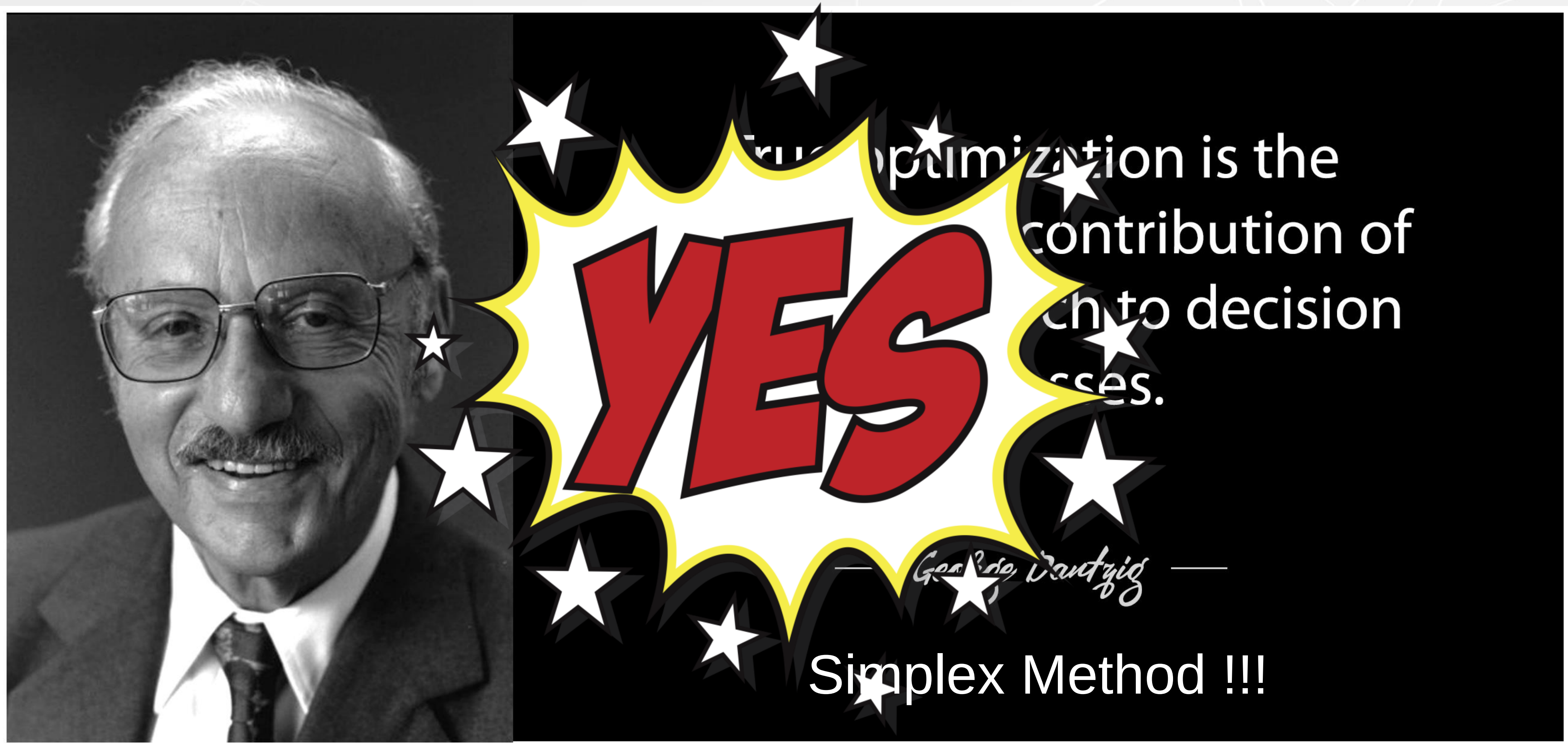
Enumeration of solutions of the system of equations for the furniture problem

Points of interest	Vertex of the polyhedron	Objective function value. Revenue = 45x1 + 80x2
(0,0)	Yes (feasible)	0 = 45*0 + 80*0
(0,20)	Yes (feasible)	1600 = 45*0 + 80*20
(24,14)	Yes (feasible)	2200 = 45*24 + 80*14 Optimal!!
(45,0)	Yes (feasible)	2025 = 45*45 + 80*0
(0,30)	No (infeasible)	2400 = 45*30 + 80*30
(80,0)	No (infeasible)	3600 = 45*80 + 80*0

$\sim 10^{88}$

This number is larger than the number of atoms ($\sim 10^{80}$) in the known universe !!

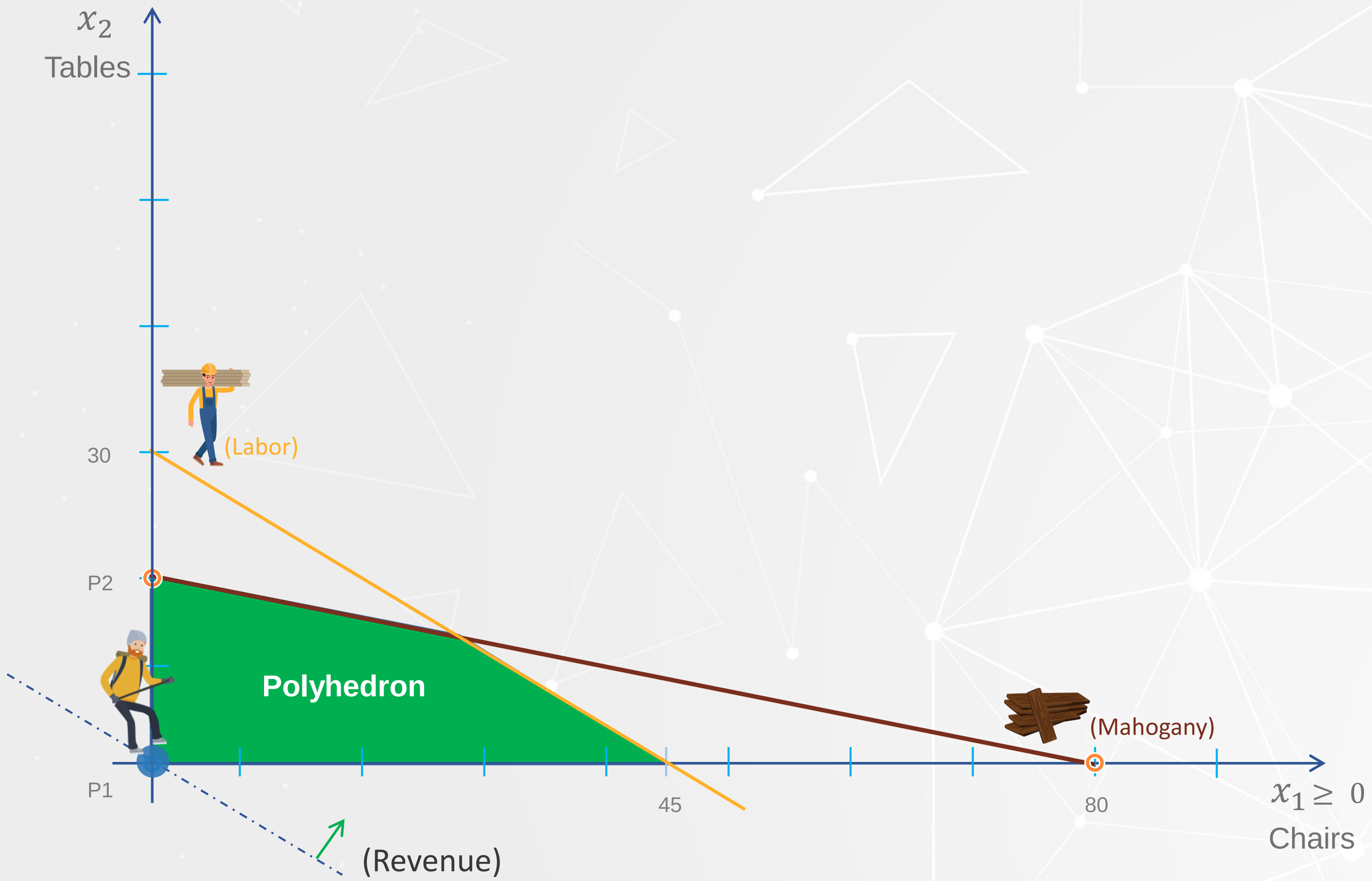
Is there a way that we can traverse vertices in the polyhedron in a more efficient way?



Simplex method

Overview

Linear Programming



Furniture problem LP problem formulation

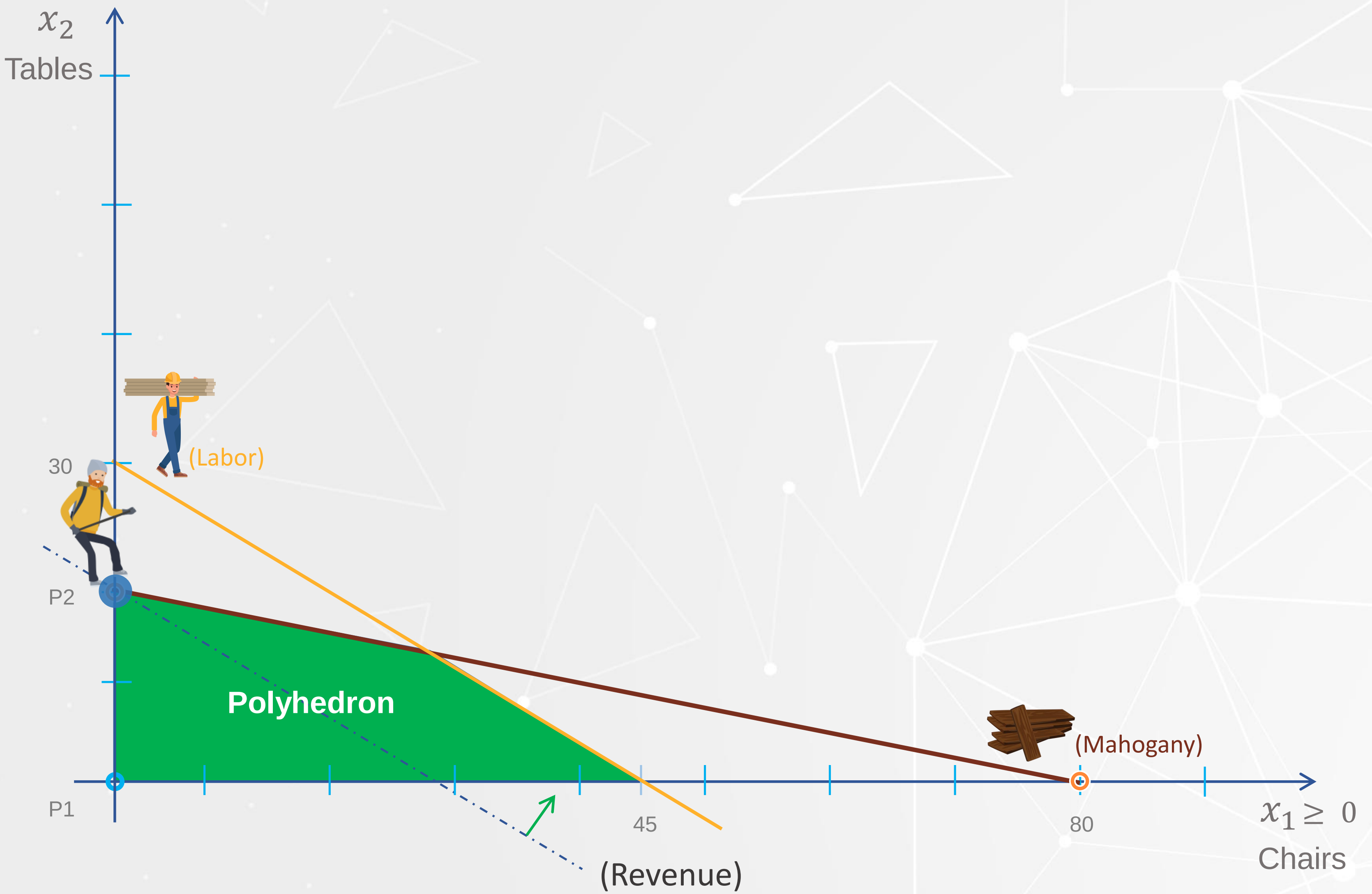
(1.0). Max revenue $= 45x_1 + 80x_2$

(2.0) $5x_1 + 20x_2 \leq 400$ Mahogany

(3.0). $10x_1 + 15x_2 \leq 450$ Labor

$\boldsymbol{x_1, x_2 \geq 0}$ Non – negativity

Linear Programming



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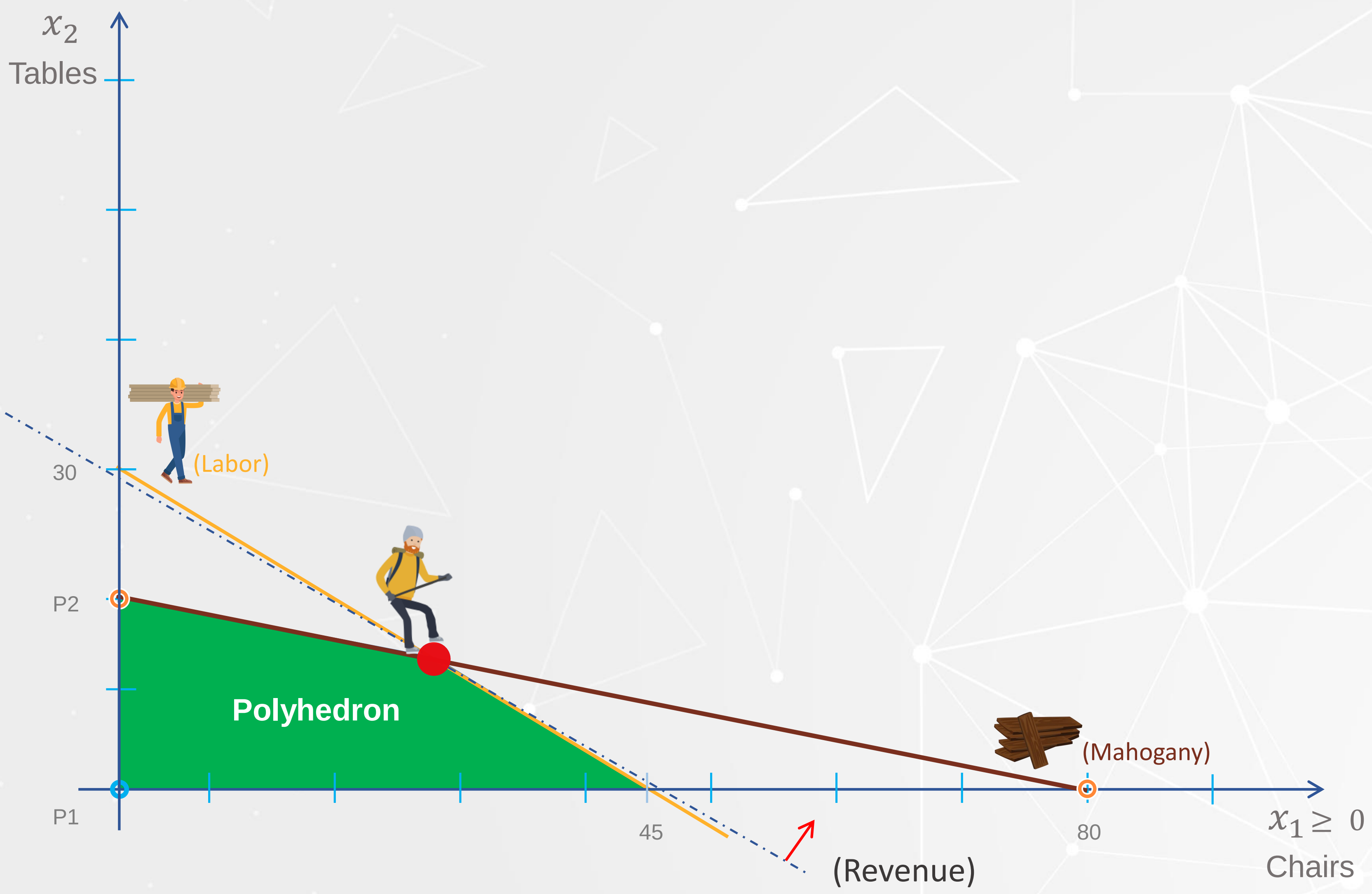
Linear Programming



Furniture problem LP problem formulation

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$\boldsymbol{x_1, x_2 \geq 0}$ Non – negativity

Linear Programming/Simplex Method

We call the formulation of an LP problem the **original LP problem**

$$(1.0).\quad \text{Max revenue} = 45x_1 + 80x_2$$

$$(2.0)\quad 5x_1 + 20x_2 \leq 400 \quad \text{Mahogany}$$

$$(3.0).\quad 10x_1 + 15x_2 \leq 450 \quad \text{Labor}$$

$$\boldsymbol{x_1, x_2 \geq 0} \quad \text{Non} - \text{negativity}$$

GUROBI OPTIMIZATION

Linear Programming/Simplex Method

The original LP problem in a **standard form** is:

$$(1.0).\quad \text{Max revenue} = 45x_1 + 80x_2$$

$$(2.0)\quad 5x_1 + 20x_2 + h_1 = 400 \quad \text{mahogany}$$

$$(3.0).\quad 10x_1 + 15x_2 + h_2 = 450 \quad \text{Labor}$$

$$\boldsymbol{x_1, x_2,} h_1, h_2 \geq \mathbf{0} \quad \text{Non} - \text{negativity}$$

Original LP problem.

$$(1.0).\quad \text{Max revenue} = 45x_1 + 80x_2$$

$$(2.0)\quad 5x_1 + 20x_2 \leq 400 \quad \text{mahogany}$$

$$(3.0).\quad 10x_1 + 15x_2 \leq 450 \quad \text{Labor}$$

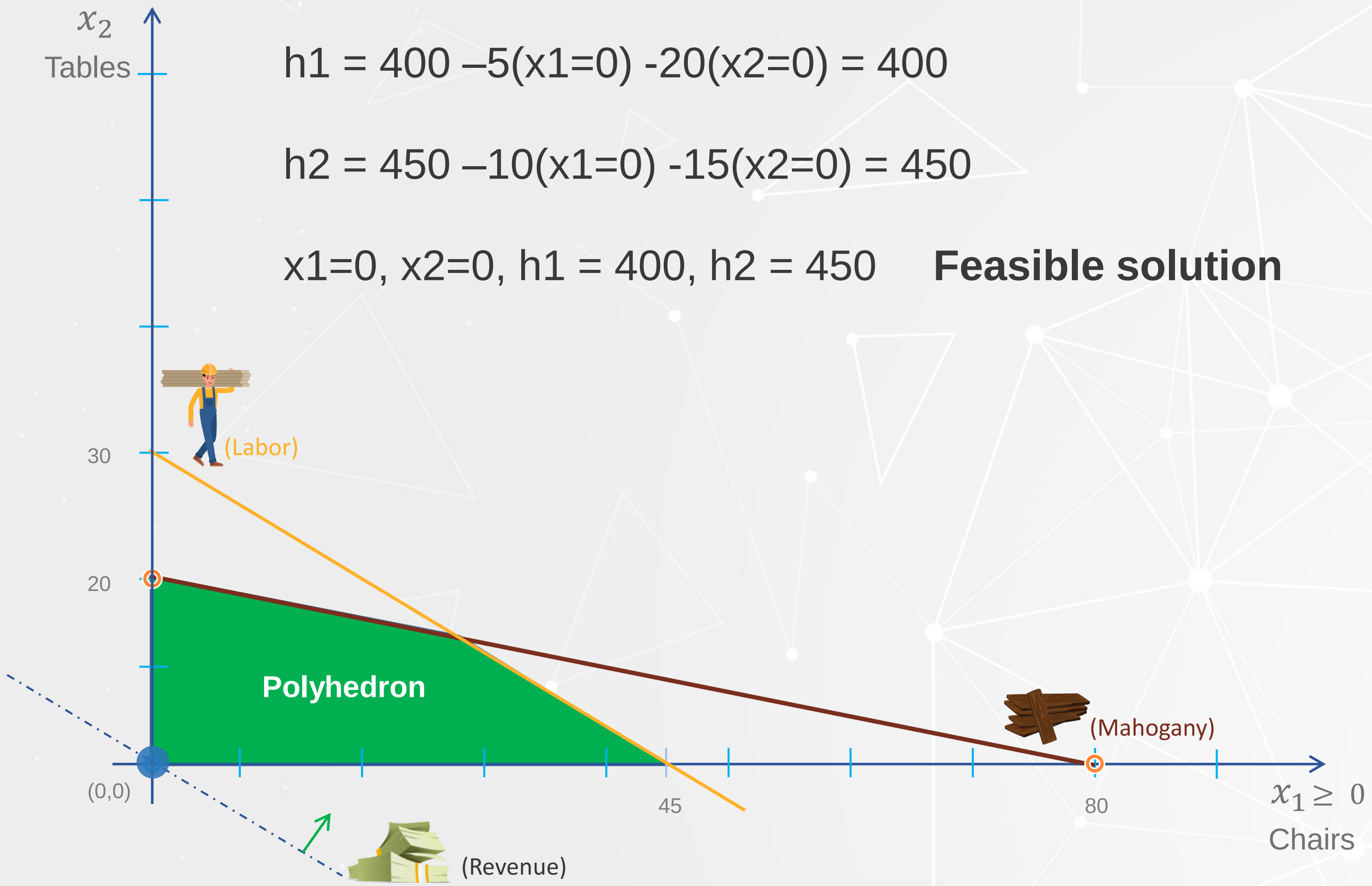
$$\boldsymbol{x_1, x_2} \geq \mathbf{0} \quad \text{Non} - \text{negativity}$$

Linear Programming/Simplex Method .. 2

h1 = 400 –5(x1=0) -20(x2=0) = 400

h2 = 450 –10(x1=0) -15(x2=0) = 450

x1=0, x2=0, h1 = 400, h2 = 450 **Feasible solution**



Furniture **problem standard form**

$$(1.0).\quad \text{Max revenue} = 45x_1 + 80x_2$$
$$(2.0)\quad 5x_1 + 20x_2 + h_1 = 400 \quad \text{Mahogany}$$
$$(3.0).\quad 10x_1 + 15x_2 + h_2 = 450 \quad \text{Labor}$$
$$\boldsymbol{x_1, x_2, h_1, h_2 \geq 0} \quad \text{Non – negativity}$$

(x1=0, x2=0) initial solution
Revenue = 0

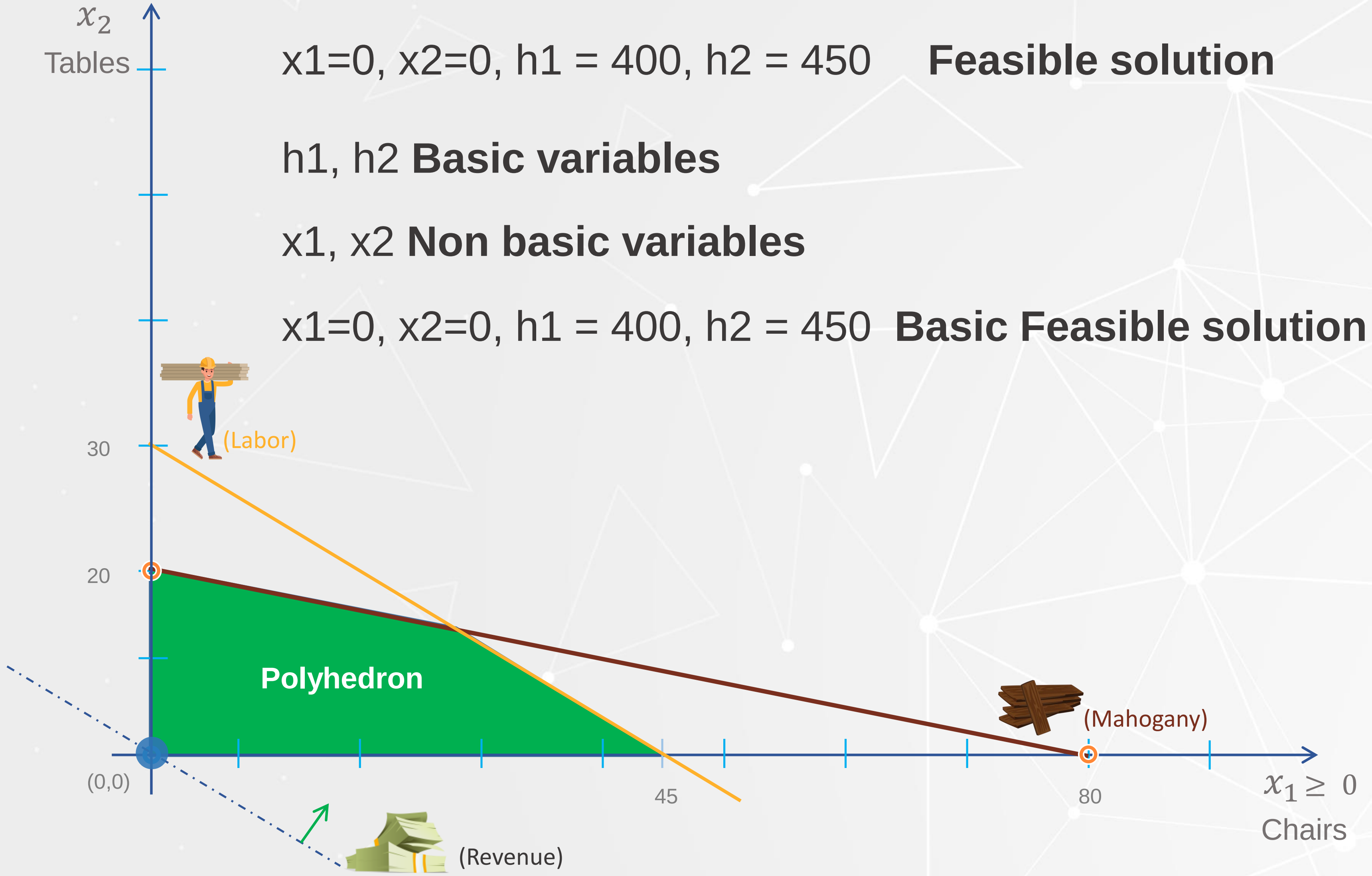
Linear Programming/Simplex Method .. 2

x1=0, x2=0, h1 = 400, h2 = 450 **Feasible solution**

h1, h2 **Basic variables**

x1, x2 **Non basic variables**

x1=0, x2=0, h1 = 400, h2 = 450 **Basic Feasible solution**



Furniture **problem standard form**

$$(1.0).\ \text{Max revenue} = 45x_1 + 80x_2$$
$$(2.0)\ \ 5x_1 + 20x_2 + h_1 = 400 \quad \text{Mahogany}$$
$$(3.0).\ \ 10x_1 + 15x_2 + h_2 = 450 \quad \text{Labor}$$
$$\boldsymbol{x_1, x_2, h_1, h_2 \geq 0} \quad \text{Non-negativity}$$

GUROBI OPTIMIZATION

Linear Programming/Simplex Method .. 3

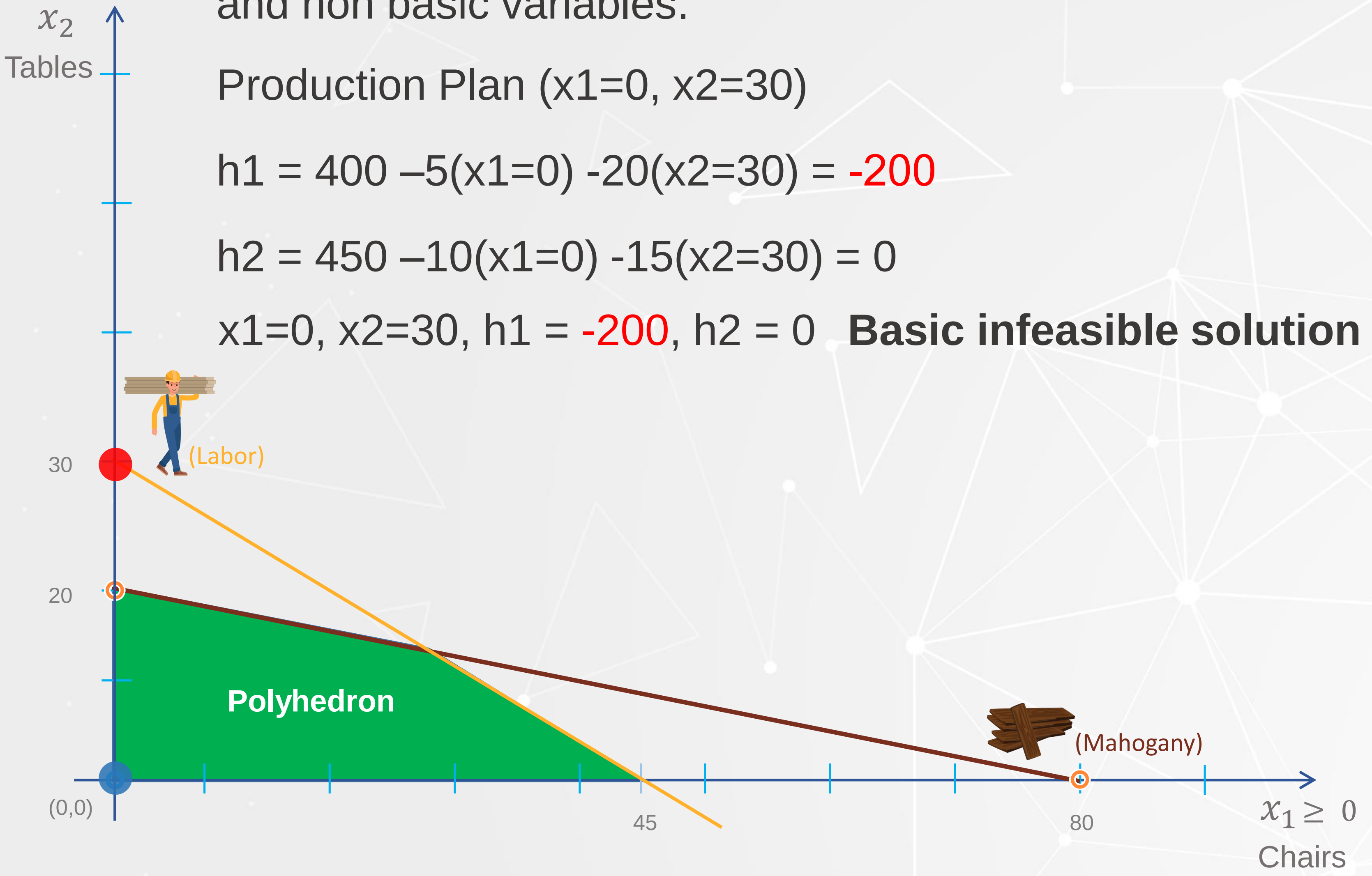
A basic solution is defined by the values of the basic and non basic variables.

Production Plan (x1=0, x2=30)

h1 = 400 –5(x1=0) -20(x2=30) = -200

h2 = 450 –10(x1=0) -15(x2=30) = 0

x1=0, x2=30, h1 = -200, h2 = 0 **Basic infeasible solution**



Furniture **problem standard** form

$$(1.0).\quad \text{Max revenue} = 45x_1 + 80x_2$$

$$(2.0)\quad 5x_1 + 20x_2 + h_1 = 400 \quad \text{Mahogany}$$

$$(3.0).\quad 10x_1 + 15x_2 + h_2 = 450 \quad \text{Labor}$$

$$\boldsymbol{x_1, x_2, h_1, h_2 \geq 0} \quad \text{Non – negativity}$$

Linear Programming/Simplex Method .. 4

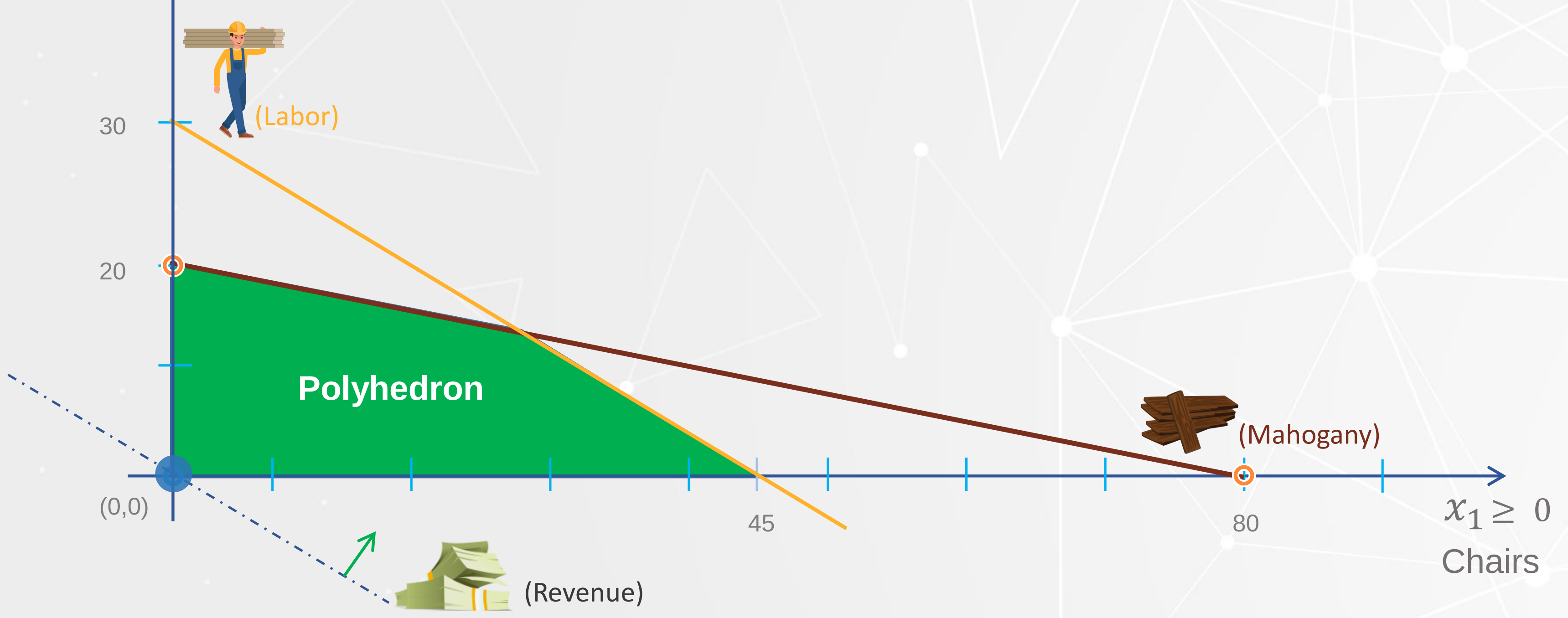
LP problem in a **canonical form** with respect to the basic variables (h1, h2) :

$$(1.0)\quad Max\ z = 45x_1 + 80x_2 + 0h_1 + 0h_2$$

$$(2.0)\quad h_1 = 400 - 5x_1 - 20x_2$$ Unused mahogany

$$(3.0)\quad h_2 = 450 - 10x_1 - 15x_2$$ Unused labor

$$x_1, x_2, h_1, h_2 \geq 0$$ Non-negativity



Furniture **problem standard form**

$$(1.0).\quad \text{Max revenue} = 45x_1 + 80x_2$$

$$(2.0)\quad 5x_1 + 20x_2 + h_1 = 400 \quad \text{Mahogany}$$

$$(3.0).\quad 10x_1 + 15x_2 + h_2 = 450 \quad \text{Labor}$$

$$\boldsymbol{x_1, x_2, h_1, h_2 \geq 0} \quad \text{Non-negativity}$$

Reduced costs:
Objective funcion coefficients of non basic variables (x1, x2)

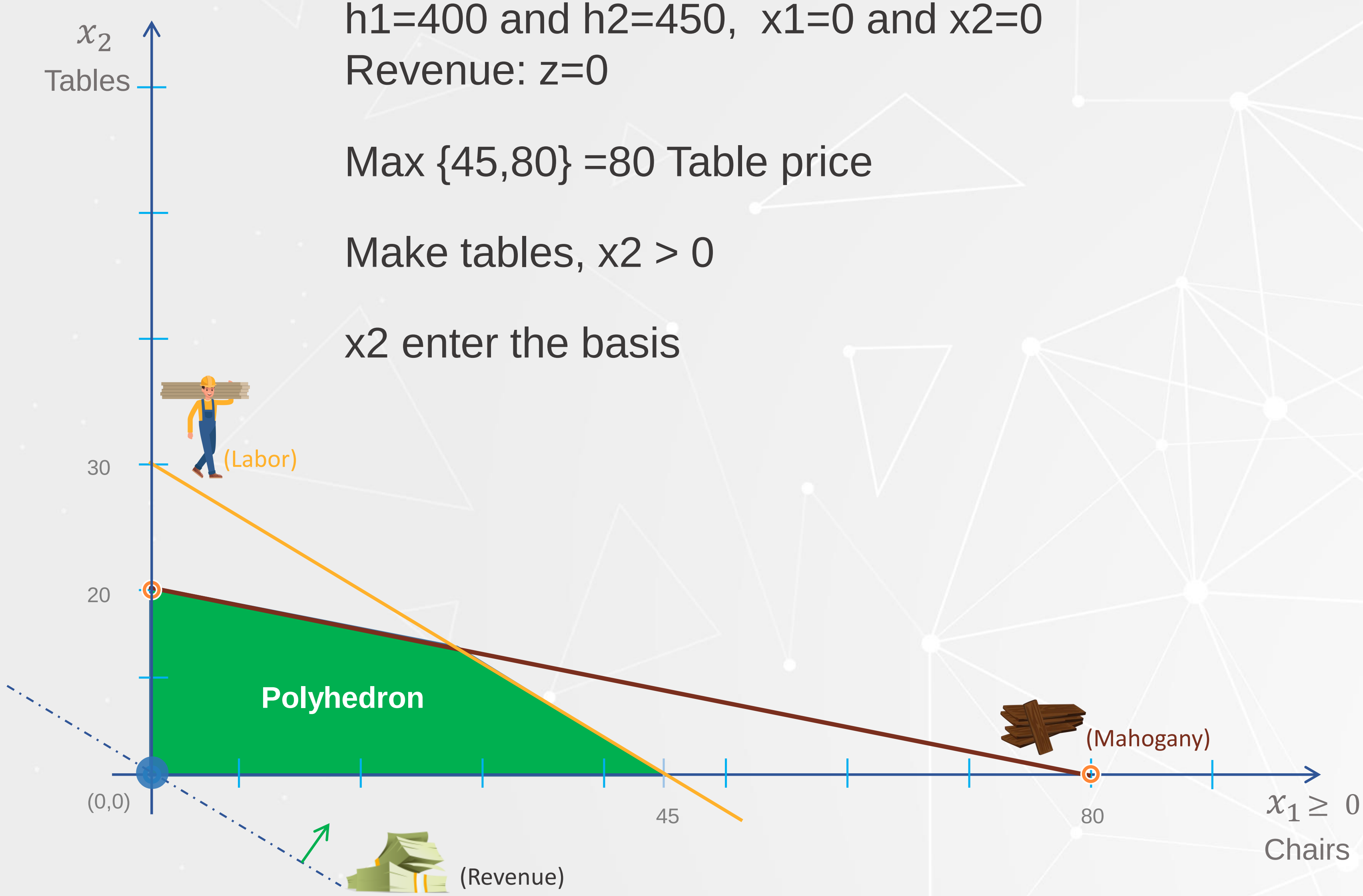
Linear Programming/Simplex Method .. 5

Current basic feasible solution:
h1=400 and h2=450, x1=0 and x2=0
Revenue: z=0

Max {45,80} =80 Table price

Make tables, x2 > 0

x2 enter the basis



Furniture problem **canonical form**

(1.0) $Max\ z = 45x_1 + 80x_2 + 0h_1 + 0h_2$

(2.0) $h_1 = 400 - 5x_1 - 20x_2$ Mahogany

(3.0) $h_2 = 450 - 10x_1 - 15x_2$ Labor

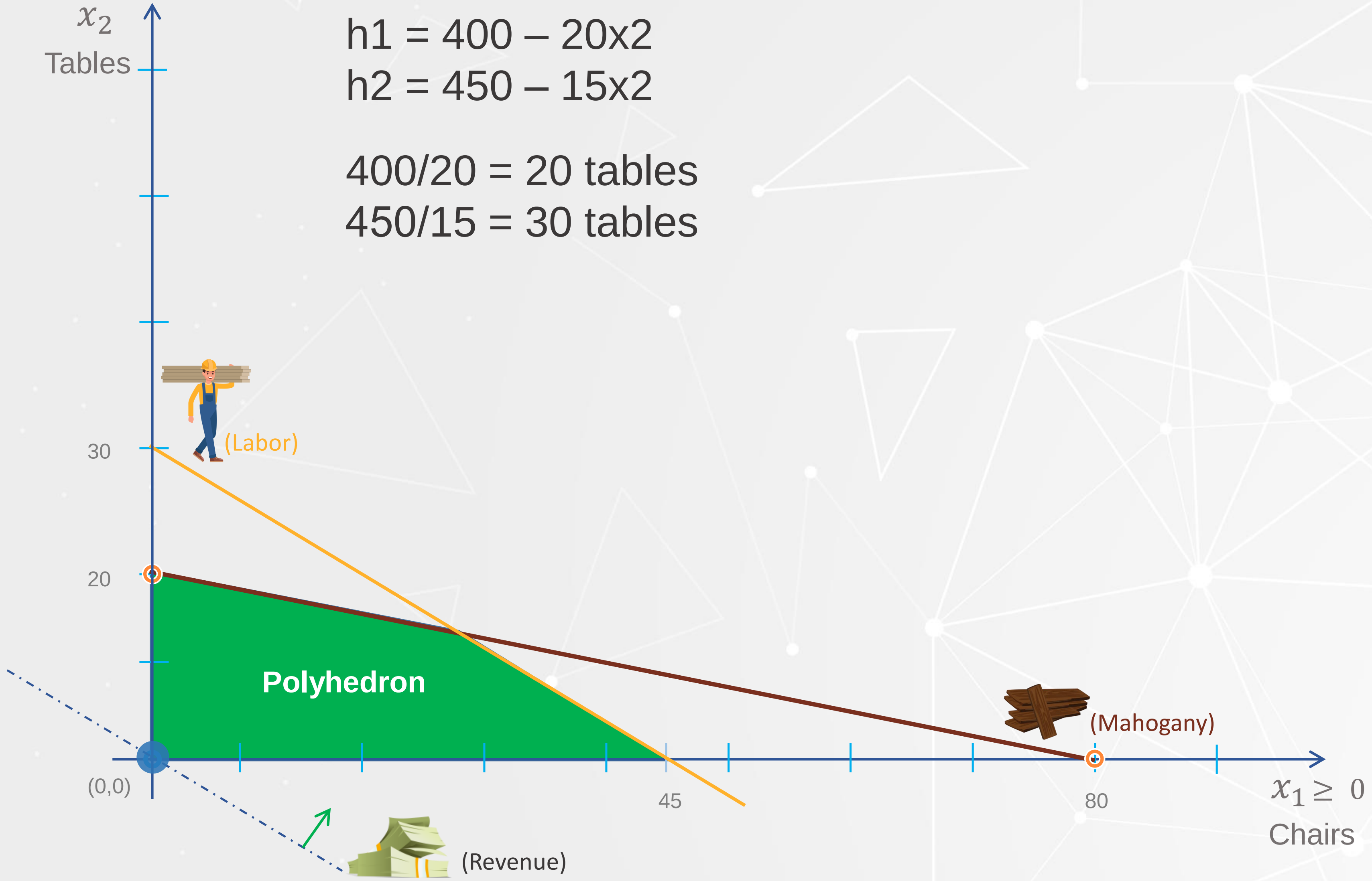
$x_1, x_2, h_1, h_2 \geq 0$ Non-negativity

Linear Programming/Simplex Method .. 5

How many tables (x2) can we make?

h1 = 400 – 20x2
h2 = 450 – 15x2

400/20 = 20 tables
450/15 = 30 tables



Furniture problem **canonical form**

(1.0) $Max\ z = 45x_1 + 80x_2 + 0h_1 + 0h_2$

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(3.0) $h_2 = 450 - 10x_1 - 15x_2$ Labor

$x_1, x_2, h_1, h_2 \geq 0$ Non-negativity

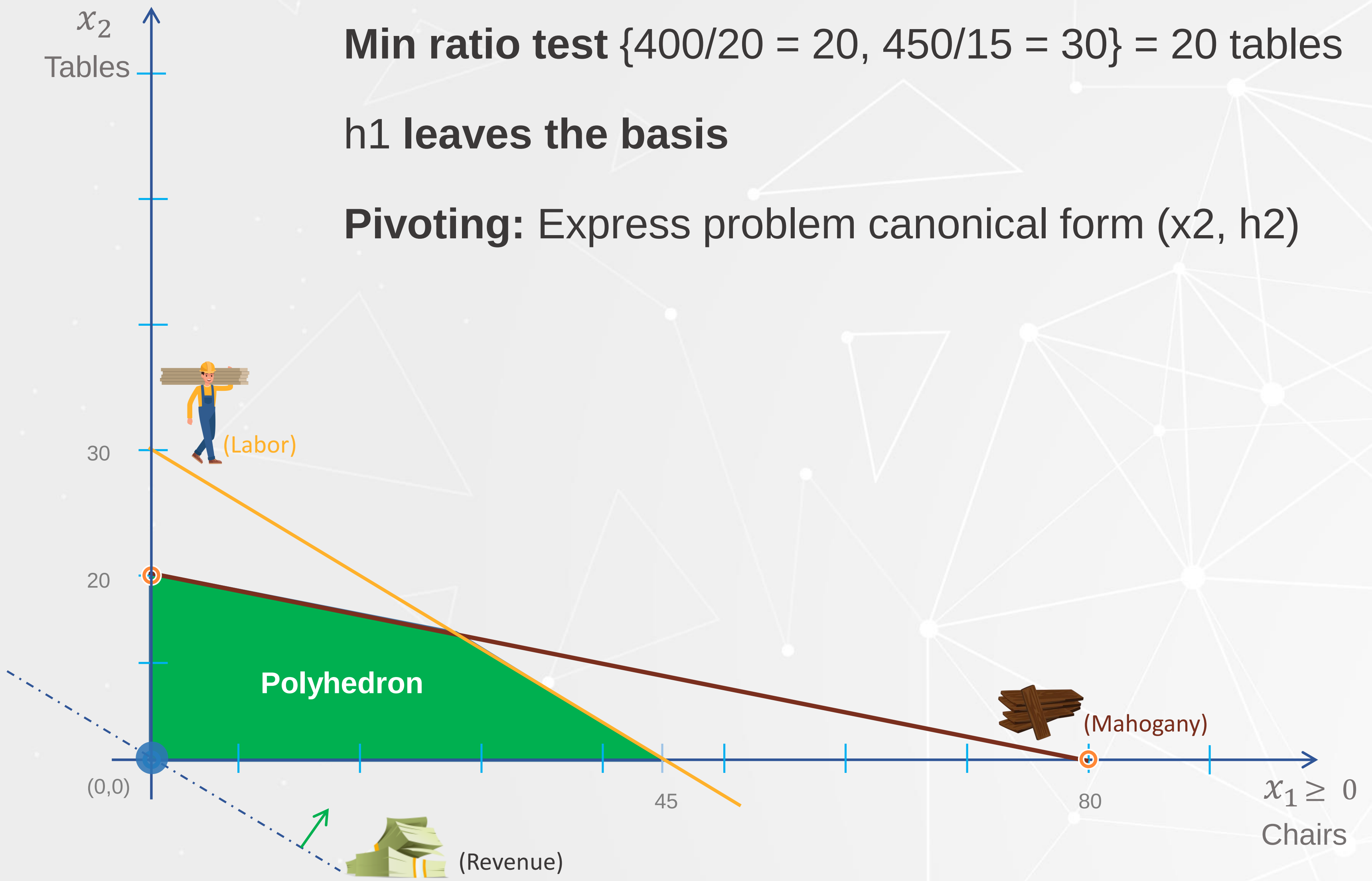
Linear Programming/Simplex Method .. 5

If x2 =30, h1 = 400 – 20(x2=30) = -200!!!!

Min ratio test {400/20 = 20, 450/15 = 30} = 20 tables

h1 **leaves the basis**

Pivoting: Express problem canonical form (x2, h2)



Furniture problem **canonical form**

(1.0) $Max\ z = 45x_1 + 80x_2 + 0h_1 + 0h_2$

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$x_1, x_2, h_1, h_2 \geq 0$ Non-negativity

GUROBI OPTIMIZATION

Linear Programming/Simplex Method (Pivoting) .. 6

$$(1.0) \quad Max\ z = 45x_1 + 80x_2 + 0h_1 + 0h_2$$

$$(2.0) \quad h_1 = 400 - 5x_1 - 20x_2$$

$$(3.0) \quad h_2 = 450 - 10x_1 - 15x_2$$

$$x_1, x_2, h_1, h_2 \geq 0$$

In equation 2.0, express x2 in terms of x1 and h1

$$(2.0) \quad x_2 = 20 - ({}^1/_4)x_1 - ({}^1/_{20})h_1$$

Linear Programming/Simplex Method (Pivoting) .. 6

In equation 2.0, express x2 in terms of x1 and h1

$$(2.0) \quad x_2 = 20 - (1/4)x_1 - (1/20)h_1$$

We substitute the value of x2 in equation (3.0)

$$(3.0) \quad h_2 = 450 - 10x_1 - 15(x_2 = 20 - (1/4)x_1 - (1/20)h_1)$$

$$= 150 - (25/4)x_1 + (3/4)h_1$$

Substitute the value of x2 in (1.0), the objective function

$$(1.0) \quad z = 45x_1 + 80(20 - (1/4)x_1 - (1/20)h_1) + 0h_1 + 0h_2$$

$$= 1600 + 25x_1 + 0x_2 - 4h_1 + 0h_2$$

$$(1.0) \quad Max\ z = 45x_1 + 80x_2 + 0h_1 + 0h_2$$

$$(2.0) \quad h_1 = 400 - 5x_1 - 20x_2$$

$$(3.0) \quad h_2 = 450 - 10x_1 - 15x_2$$

$$x_1, x_2, h_1, h_2 \geq 0$$

Linear Programming/Simplex Method .. 7

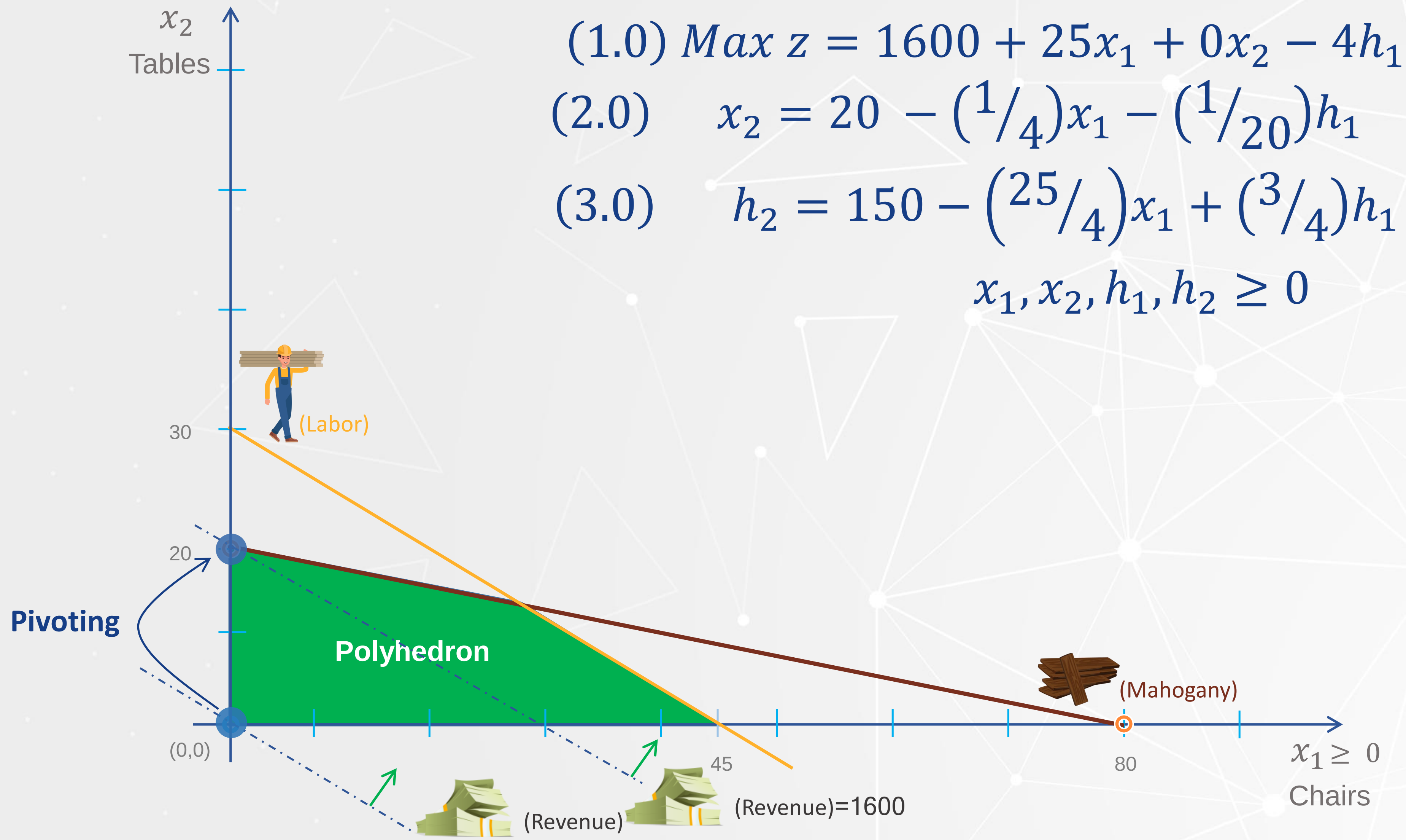
Furniture LP problem in a canonical form with respect to the basic variables (x2, h2).

$$(1.0)\ Max\ z = 1600 + 25x_1 + 0x_2 - 4h_1 + 0h_2$$

$$(2.0)\quad x_2 = 20 - (1/4)x_1 - (1/20)h_1$$ Production of tables

$$(3.0)\quad h_2 = 150 - (25/4)x_1 + (3/4)h_1$$ Unused labor capacity

$$x_1, x_2, h_1, h_2 \geq 0$$ Non-negativity



GUROBI OPTIMIZATION

Linear Programming/Simplex Method (Pivoting) .. 8

$$(1.0)\ Max\ z = 1600 + 25x_1 + 0x_2 - 4h_1 + 0h_2$$

$$(2.0)\quad x_2 = 20 - \left({}^{1}/_{4}\right)x_1 - \left({}^{1}/_{20}\right)h_1$$ Production of tables

$$(3.0)\quad h_2 = 150 - \left({}^{25}/_{4}\right)x_1 + \left({}^{3}/_{4}\right)h_1$$ Unused labor capacity

$$x_1, x_2, h_1, h_2 \geq 0$$ Non-negativity

Simplex method: **iteration 2**

Step1: x1 enters the basis.

Step2: minimum ratio test, min{20/(1/4)=80 ,150/(25/4)=24 }=24. h2 leaves the basis

Step3: Pivoting express problem in canonical form with respect to (x2, x1)

GUROBI OPTIMIZATION

Linear Programming/Simplex Method (Pivoting) .. 9

In equation 3.0, express x1 in terms of h1 and h2

$$(3.0)\quad x_1 = 24 + (3/25)h_1 - (4/25)h_2$$

We substitute the value of x1 in equation (2.0)

$$(2.0)\quad x_2 = 14 - \left({}^{2}/_{25}\right)h_1 + \left({}^{1}/_{25}\right)h_2$$

Substitute the value of x1 in (1.0), the objective function

$$z = 2200 + 0x_1 + 0x_2 - 1h_1 - 4h_2$$

$$(1.0)\ Max\ z = 1600 + 25x_1 + 0x_2 - 4h_1 + 0h_2$$

$$(2.0)\quad x_2 = 20 - \left({}^{1}/_{4}\right)x_1 - \left({}^{1}/_{20}\right)h_1$$

$$(3.0)\quad h_2 = 150 - \left({}^{25}/_{4}\right)x_1 + \left({}^{3}/_{4}\right)h_1$$

$$x_1, x_2, h_1, h_2 \geq 0$$

Linear Programming/Simplex Method (Pivoting) .. 10

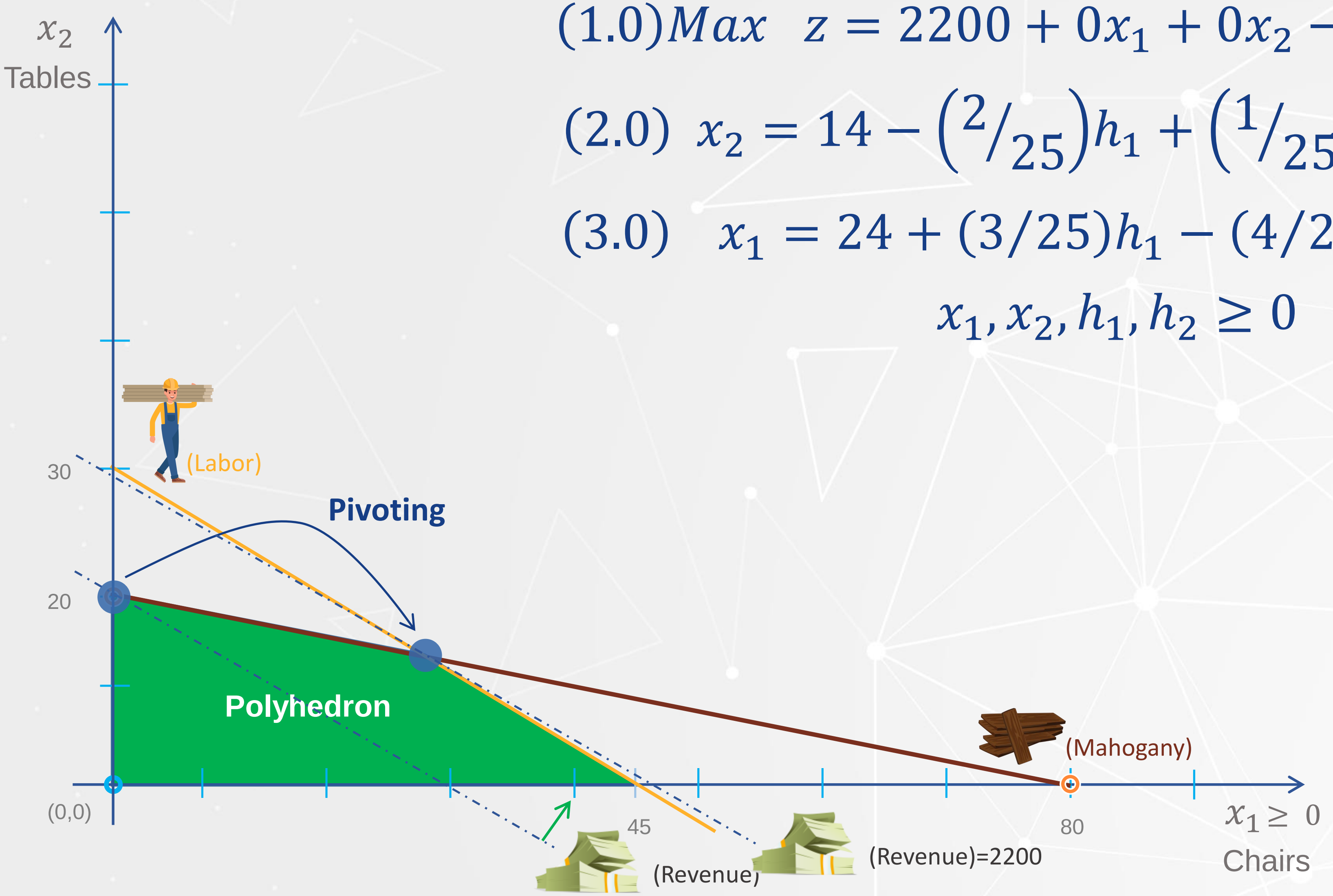
Furniture LP problem in a canonical form with respect to the basic variables (x2, x1).

$$(1.0)\ Max\quad z = 2200 + 0x_1 + 0x_2 - 1h_1 - 4h_2$$

$$(2.0)\quad x_2 = 14 - \left({}^{2}/_{25}\right)h_1 + \left({}^{1}/_{25}\right)h_2$$ Production of tables

$$(3.0)\quad x_1 = 24 + (3/25)h_1 - (4/25)h_2$$ Production of chairs

$$x_1, x_2, h_1, h_2 \geq 0$$ Non-negativity

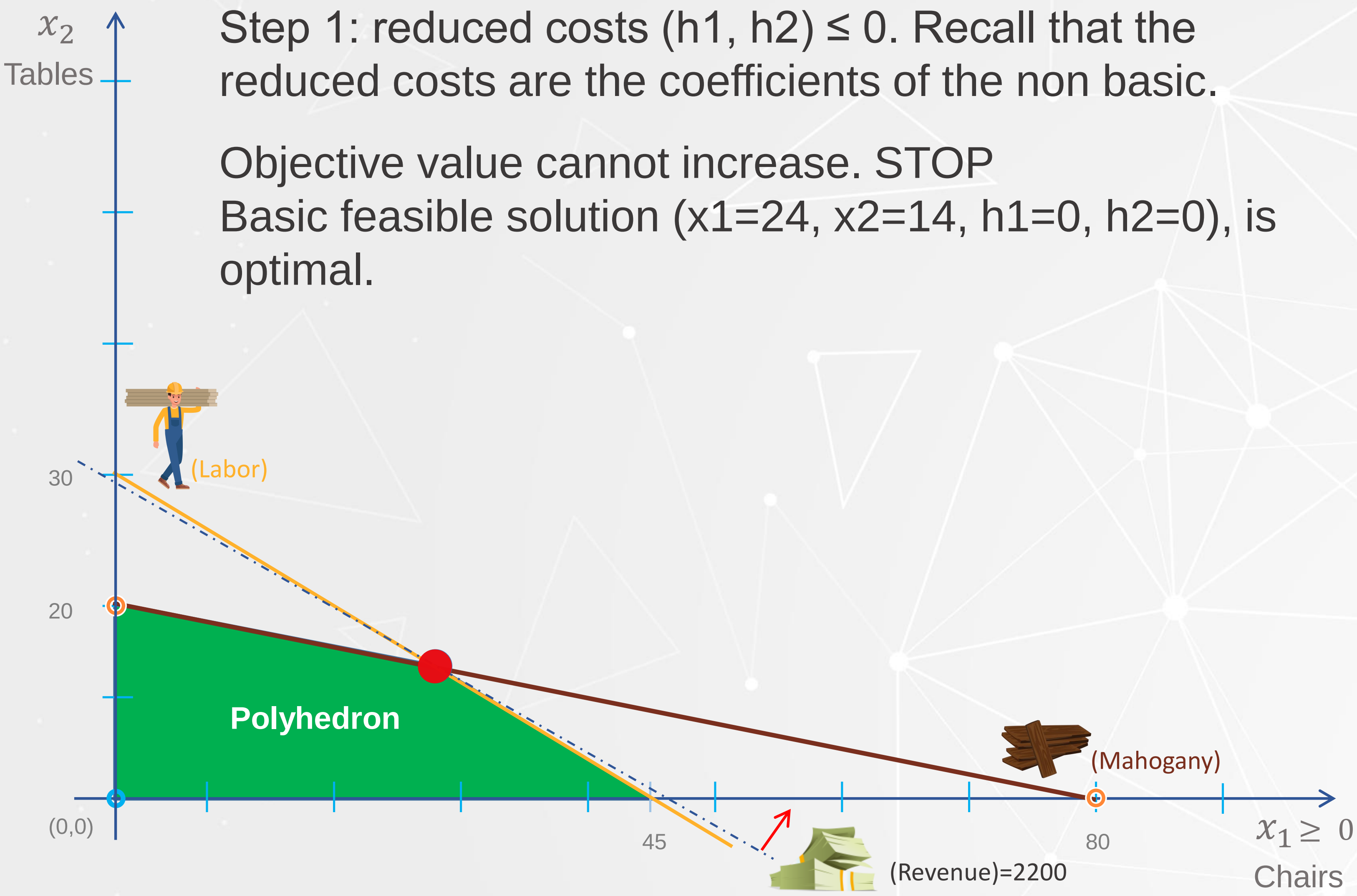


Linear Programming/Simplex Method.. 11

Simplex method: iteration 3.

Step 1: reduced costs (h1, h2) ≤ 0. Recall that the reduced costs are the coefficients of the non basic.

Objective value cannot increase. STOP
Basic feasible solution (x1=24, x2=14, h1=0, h2=0), is optimal.



$$(1.0)\, Max \quad z = 2200 + 0x_1 + 0x_2 - 1h_1 - 4h_2$$

$$(2.0) \quad x_2 = 14 - \left({}^{2}/_{25}\right)h_1 + \left({}^{1}/_{25}\right)h_2$$

$$(3.0) \quad x_1 = 24 + (3/25)h_1 - (4/25)h_2$$

$$x_1, x_2, h_1, h_2 \geq 0$$

GUROBI OPTIMIZATION

Summary of simplex method for the maximization case

1. Transform the original LP problem into the standard form. Consider an initial basic feasible solution.

2. Express the LP problem in a canonical form with respect to the current basic feasible solution.

3. If the reduced costs of all the non basic variables are ≤ 0, **STOP** –the current basic feasible solution is optimal. Else, choose a non basic variable with the largest positive reduced cost to enter the basis.

4. Consider the column vector of the non basic variable entering the basis. If all the coefficients of this column vector are positive, the entering non basic variable can be arbitrarily large, hence the LP problem is unbounded.
    i. Assume that the column vector has at least one negative component.
    ii. Apply the minimum ratio test over the equations where the entering non basic variable has negative coefficients to determine the basic variable that will leave the basis.
    iii. (Pivoting) Go to 2.) to determine the new basic solution.