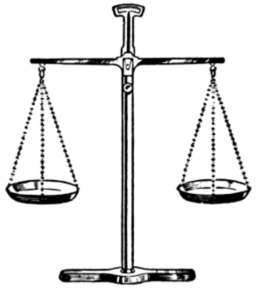


Integer Programming

Introduction

- When one or more variables in an LP problem must assume an integer value we have an Integer Linear Programming (ILP) problem.
- ILPs occur frequently…
    - Scheduling workers
    - Manufacturing airplanes
- Integer variables also allow us to build more accurate models for a number of common problems.

Relaxation

- Original ILP

  MAX: $2X_1 + 3X_2$

  S.T.: $X_1 + 3X_2 <= 8.25$

  $2.5X_1 + X_2 <= 8.75$

  $X_1, X_2 >= 0$

  X_1, X_2 must be integers

- LP Relaxation

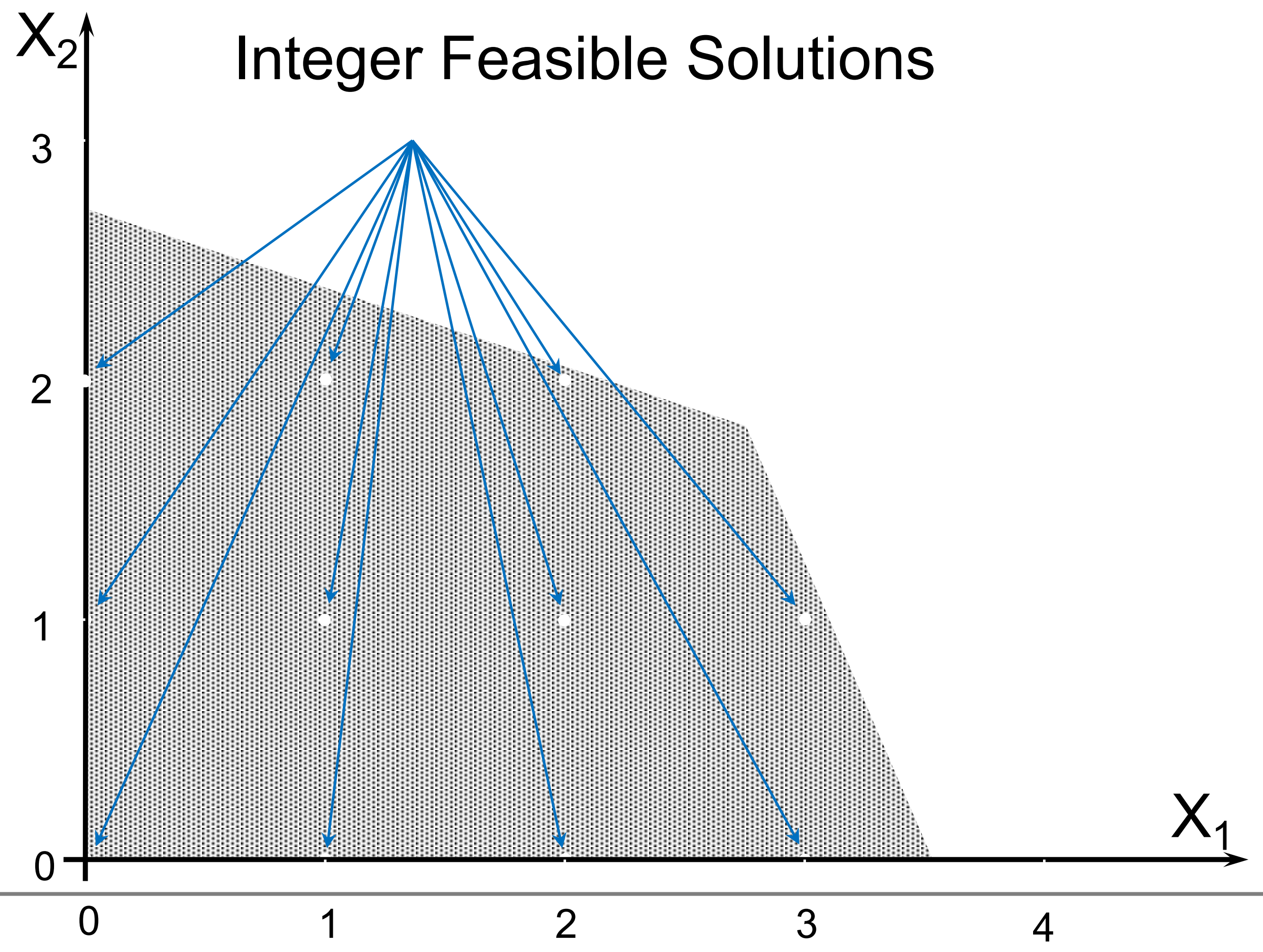
  MAX: $2X_1 + 3X_2$

  S.T.: $X_1 + 3X_2 <= 8.25$

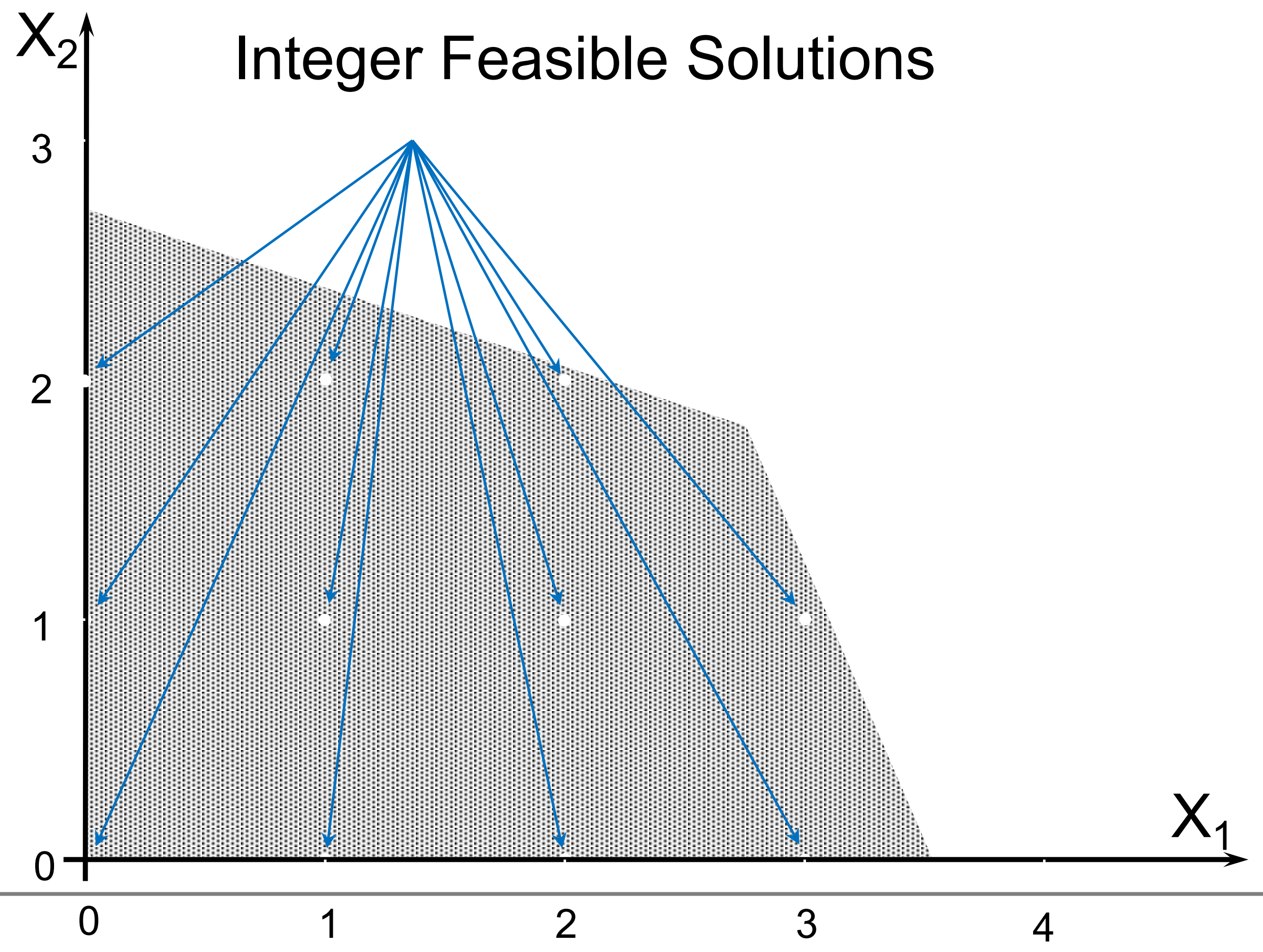
  $2.5X_1 + X_2 <= 8.75$

  $X_1, X_2 >= 0$

Integer Feasible vs. LP Feasible Region



Integer Feasible vs. LP Feasible Region



Integrality Conditions

MAX: $350X_1 + 300X_2$ } profit

S.T.: $1X_1 + 1X_2 <= 200$ } pumps

$9X_1 + 6X_2 <= 1566$ } labor

$12X_1 + 16X_2 <= 2880$ } tubing

$X_1, X_2 >= 0$ } nonnegativity

X_1, X_2 must be integers } integrality

Integrality conditions are easy to state but make the problem much more difficult (and sometimes impossible) to solve.

Solving ILP Problems

- When solving an LP relaxation, sometimes you "get lucky" and obtain an integer feasible solution.
- This was the case in the original Blue Ridge Hot Tubs problem in earlier chapters.
- But what if we reduce the amount of labor available to 1520 hours and the amount of tubing to 2650 feet?

Integrality Conditions

MAX: $350X_1 + 300X_2$ } profit

S.T.: $1X_1 + 1X_2 <= 200$ } pumps

$9X_1 + 6X_2 <= 1520$ } labor

$12X_1 + 16X_2 <= 2650$ } tubing

$X_1, X_2 >= 0$ } nonnegativity

X_1, X_2 must be integers } integrality

Optimal solution of the relaxed problem:

$X_1 = 116{,}9444$ $X_2 = 77{,}9167$

Corresponding to a maximum profit of $64306

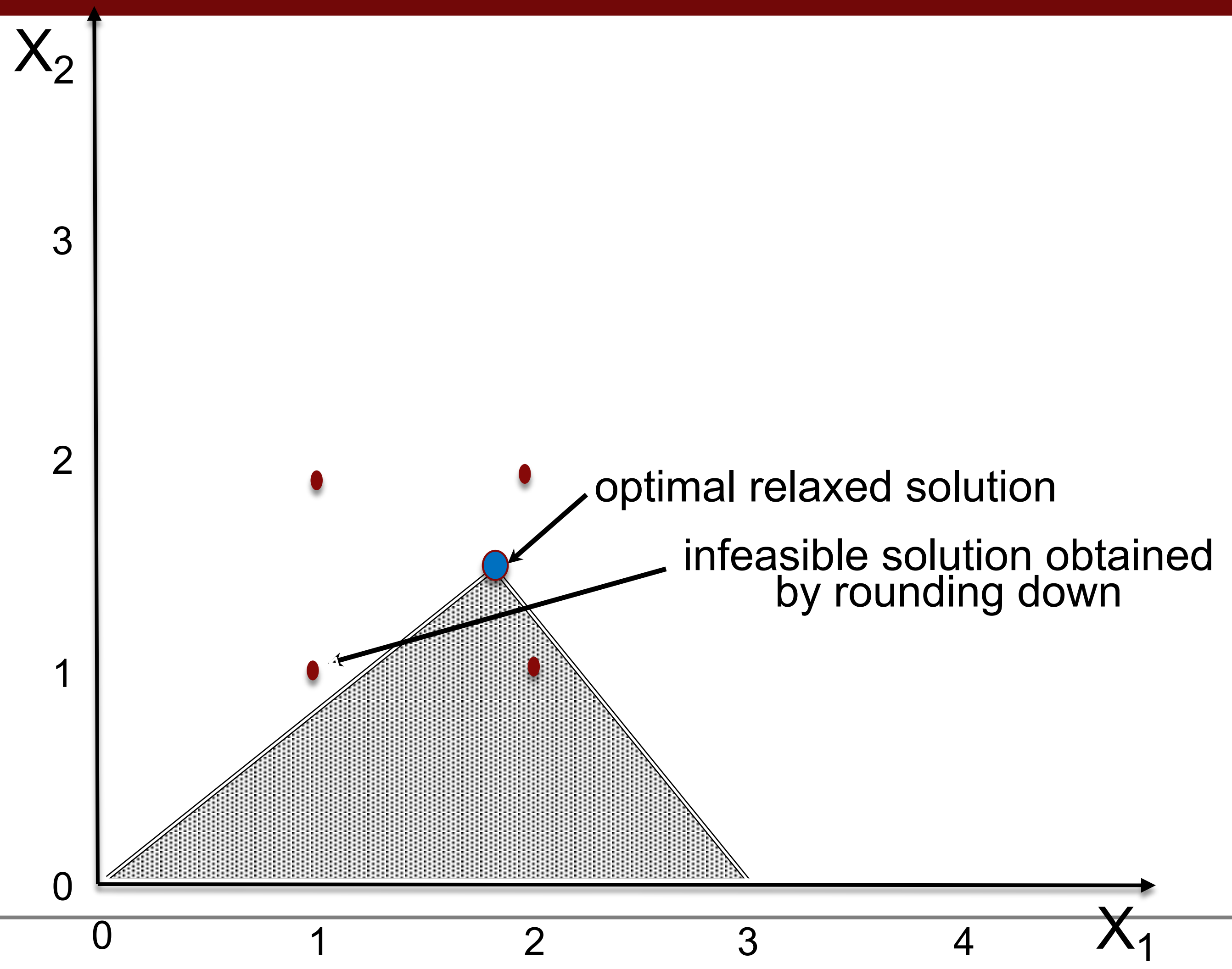
Rounding

- It is tempting to simply round a fractional solution to the closest integer solution.
- In general, this does not work reliably:
  - The rounded solution may be infeasible.
  - The rounded solution may be suboptimal.

How Rounding Down Can Result in an Infeasible Solution

	Blue Ridge Hot Tubs			
	Aqua-Spas	Hydro-Luxes		
Number to Make	117	78	Total Profit	
Unit Profits	$350	$300	$64,350	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1521	1520
Tubing Req'd	12	16	2652	2650

How Rounding Down Can Result in an Infeasible Solution



How Rounding Down Can Result in an Infeasible Solution

	Blue Ridge Hot Tubs			
	Aqua-Spas	Hydro-Luxes		
Number to Make	116	77	Total Profit	
Unit Profits	$350	$300	$63,700	
Constraints			Used	Available
Pumps Req'd	1	1	193	200
Labor Req'd	9	6	1506	1520
Tubing Req'd	12	16	2624	2650

Stopping Rules

- Because B&B can take so long, most ILP packages allow you to specify a suboptimality tolerance factor.
- This allows you to stop once an integer solution is found that is within some % of the global optimal solution.
- Bounds obtained from LP relaxations are helpful here.
  - Example
    - LP relaxation has an optimal obj. value of $64,306.
    - 95% of $64,306 is $61,090.
    - Thus, an integer solution with obj. value of $61,090 or better must be within 5% of the optimal solution.

Bounds

- The optimal solution to an LP relaxation of an ILP problem gives us a *bound* on the optimal objective function value.

- For maximization problems, the optimal relaxed objective function values is an *upper bound* on the optimal integer value.

- For minimization problems, the optimal relaxed objective function values is a *lower bound* on the optimal integer value.

Branch-and-Bound

- The Branch-and-Bound (B&B) algorithm can be used to solve ILP problems.
- Requires the solution of a series of LP problems termed "candidate problems".
- *Theoretically*, this can solve any ILP.
- *Practically*, it often takes *LOTS* of computational effort (and time).