Optimality conditions

Linear Programming

Optimality conditions in linear programming .. 1

Consider the Furniture primal and dual problems:

Primal

(1.0) $Max\ revenue = 45x_1 + 80x_2$

(2.0) $5x_1 + 20x_2 \leq 400$ Mahogany

(3.0) $10x_1 + 15x_2 \leq 450$ Labor

$x_1, x_2 \geq 0$

Chairs Tables

Dual

(4.0) $Min\ Cost = 400w_1 + 450w_2$

(5.0) $5w_1 + 10w_2 \geq 45$ Chairs

(6.0) $20w_1 + 15w_2 \geq 80$ Tables

$w_1, w_2 \geq 0$

Mahogany Labor

Furniture primal and dual problems in standard form

(1.0) $Max\ revenue = 45x_1 + 80x_2$

(2.0) $5x_1 + 20x_2 + h_1 = 400$ Mahogany

(3.0) $10x_1 + 15x_2 + h_2 = 450$ Labor

$x_1, x_{2,} h_{1,} h_2 \geq 0$

Chairs Tables

(4.0) $Min\ Cost = 400w_1 + 450w_2$

(5.0) $5w_1 + 10w_2 - s_1 = 45$ Chairs

(6.0) $20w_1 + 15w_2 - s_2 = 80$ Tables

$w_1, w_2 s_1, s_2 \geq 0$

Mahogany Labor

h_1 is the slack variable of the mahogany constraint

h_2 is the slack variable of the labor constraint

s_1 is the surplus variable of the chairs constraint

s_2 is the surplus variable of the tables constraint

Optimality conditions in linear programming .. 2

Consider the optimal solution of both problems, primal and dual

Primal optimal solution:

x1 = 24, x2 = 14, h1 = 0, h2 = 0

$$(1.0)\ Max\ revenue = 45(x1 = 24) + 80(x2 = 14) = 2200$$

Mahogany

$$(2.0)\ 5(x1 = 24) + 20(x2 = 14) + (h1 = 0) = 400$$ binding

Labor

$$(3.0)\ 10(x1 = 24) + 15(x2 = 14) + (h2 = 0) = 450$$ binding

$$x_1, x_2, h_1, h_2 \geq 0$$

Chairs Tables

Dual optimal solution:

w1 = 1, w2 = 4, s1 = 0, s2 = 0

$$(4.0)\ Min\ Cost = 400(w1 = 1) + 450(w2 = 4) = 2200$$

Chairs

$$(5.0)\ 5(w1 = 1) + 10(w2 = 4) - (s1 = 0) = 45$$

Tables

$$(6.0)\ 20(w1 = 1) + 15(w2 = 4) - (s2 = 0) = 80$$

$$w_1, w_2 s_1, s_2 \geq 0$$

Mahogany Labor

Remarks

- Note that the mahogany and labor constraints are binding, i.e. the slack variables (h1 = h2 = 0).
- Note that the shadow price of the mahogany and labor constraints are positive, i.e. (w1 = 1, w2 = 4).
- Note that the chairs and tables constraints are binding, i.e. the surplus variables (s1 = s2 = 0).
- Note that the “shadow price” of the chairs and tables constraints are positive, i.e. (x1 = 24, x2 = 14).

Optimality conditions in linear programming .. 3

In summary

Primal optimal solution	Dual optimal solution
Chair variable x1 = 24	Chair constraint (is binding) surplus variable s1 = 0
Table variable x2 = 14	Table constraint (is binding) surplus variable s2 = 0
Mahogany constraint (is binding) slack variable h1 = 0	Mahogany shadow price w1 = 1
Labor constraint (is binding) slack variable h2 = 0	Labor shadow price w2 = 4

Complementary (a.k.a. orthogonality) conditions in linear programming optimal solutions

- x*s = 0. At optimality, the product of the decision variables in the primal problem and the associate surplus (slack) variables in the dual problem is always zero.
- h*w = 0. At optimality, the product of the slack (surplus) variables in the primal problem and the associate shadow prices in the dual problem is always zero.

Summary of optimality conditions for linear programming

- A solution (x1=24, x2=14) to the primal problem and a solution (w1=1, w2=4) of the dual problem are optimal, if and only if:

- Primal feasibility: we have found a solution of the primal problem that satisfy all its constraints.
  - The production plan satisfies the mahogany and labor constraints.

$$(2.0)\ 5(x1 = 24) + 20(x2 = 14) = 400$$ Mahogany capacity

$$(3.0)\ 10(x1 = 24) + 15(x2 = 14) = 450$$ Labor capacity

- Dual feasibility: we have found a solution of the dual problem that satisfy all its constraints.
  - The shadow prices associated to the mahogany and labor resources satisfy the price constraints for the chairs and the tables.

$$(5.0)\ 5(w1 = 1) + 10(w2 = 4) = 45$$ Chair price

$$(6.0)\ 20(w1 = 1) + 15(w2 = 4) = 80$$ Table price

Summary of optimality conditions for linear programming

- Complementary (orthogonality) conditions:

- The product of the decision variables in the primal problem and the associate surplus variables in the dual problem is always zero. (**Cost efficient**).

  - Since we are building 24 chairs, the surplus variable of the constraint of the price of chairs is 0. That is, x1*s1=0. This means that the opportunity costs of building chairs is equal to the price of the chair:

$$(5.0)\ 5(w1 = 1) + 10(w2 = 4) - (s1 = 0) = 45$$

  - Since we are building 14 tables, the surplus variable of the constraint of the price of tables is 0. That is, x2*s2=0. This means that the opportunity costs of building tables is equal to the price of the table:

$$(6.0)\ 20(w1 = 1) + 15(w2 = 4) - (s2 = 0) = 80$$

Summary of optimality conditions for linear programming

- The product of the decision variables in the dual problem and the associate slack variables in the primal problem is always zero. **(Resource efficient).**

  - Since the shadow price (opportunity cost) of mahogany is $1, the slack variable of the mahogany constraint is 0. This means we are using the mahogany resource efficiently and there is no waste.

$$(2.0)\ \ 5(x1 = 24) + 20(x2 = 14) + (h1 = 0) = 400$$

  - Since the shadow price (opportunity cost) of labor is $4, the slack variable of the labor constraint is 0. This means we are using the labor resource efficiently and there is no waste.

$$(3.0)\ \ 10(x1 = 24) + 15(x2 = 14) + (h2 = 0) = 450$$

Summary of optimality conditions for linear programming

- The optimal objective function value of the primal problem = the optimal objective function of the dual problem. This mathematical theorem is known as **strong duality**.

$$(1.0)\ Max\ revenue = 45(x1 = 24) + 80(x2 = 14) = 2200$$

$$(4.0)\ Min\ Cost = 400(w1 = 1) + 450(w2 = 4) = 2200$$