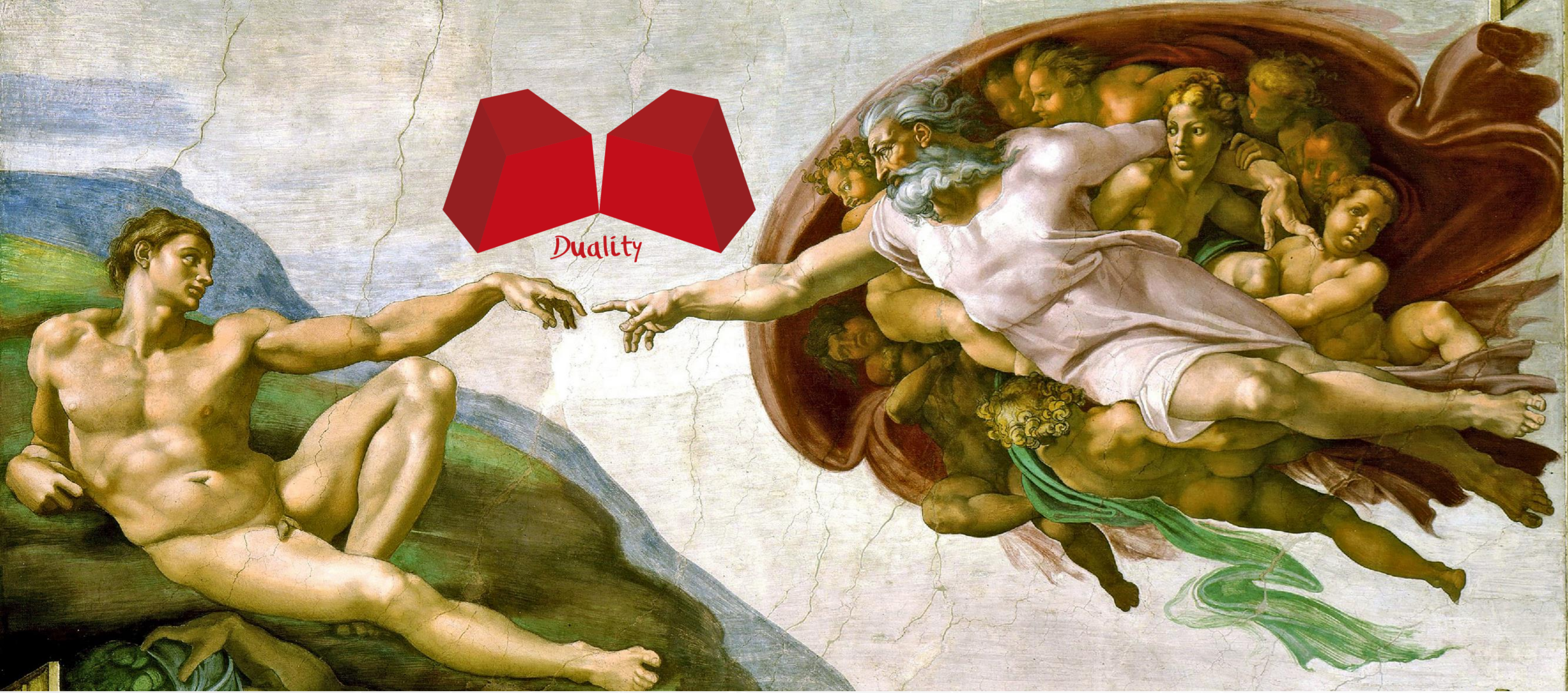


Duality in Linear Programming

GUROBI OPTIMIZATION

Linear Programming Duality

In the section of sensitivity analysis of Linear Programming, we asked the following question:

- For the furniture problem, is it profitable to make a third product, like desks?
  - Assume that the price of the desk is $110,
  - and the desk consumes 15 units of mahogany and 25 units of labor.

- Shadow prices determine the marginal worth of an additional unit of a resource:
  - The shadow price of the mahogany capacity constraint is $1.
  - The shadow price of the labor capacity constraint is $4.

- Let's compute the opportunity cost of making one desk and compare it with the price of a desk. If this opportunity cost is greater than the desk price, then it is not worth it to make desks.
  - The opportunity cost can be computed by multiplying the units of mahogany capacity that one desk built consumes by the shadow price of mahogany capacity, and multiplying the units of labor capacity that one desk built consumes by the shadow price of labor capacity:
    - That is, ($1)*15 (units of mahogany) + ($4)*25 (hours of labor) = $115 > $110.

- Therefore, investing resources to produce desks, otherwise used to produce chairs and tables, is not profitable.

GUROBI OPTIMIZATION

Linear Programming Duality .. 2

Duality in Linear Programming is an unifying theory that established the relation between an LP problem – called **Primal Problem**, and another related LP problem –called **Dual Problem,** where its decision variables (**dual variables**) are the shadow prices of the resource constraints.

Furniture Problem: Primal and Dual problems

The primal and dual of the Furniture problem are:

Primal

(1.0) $Max\ revenue = 45x_1 + 80x_2$

(2.0) $5x_1 + 20x_2 \leq 400$ Mahogany

(3.0) $10x_1 + 15x_2 \leq 450$ Labor

$x_1, x_2 \geq 0$

Chairs Tables

Dual

(4.0) $Min\ Cost = 400w_1 + 450w_2$

(5.0) $5w_1 + 10w_2 \geq 45$ Chairs

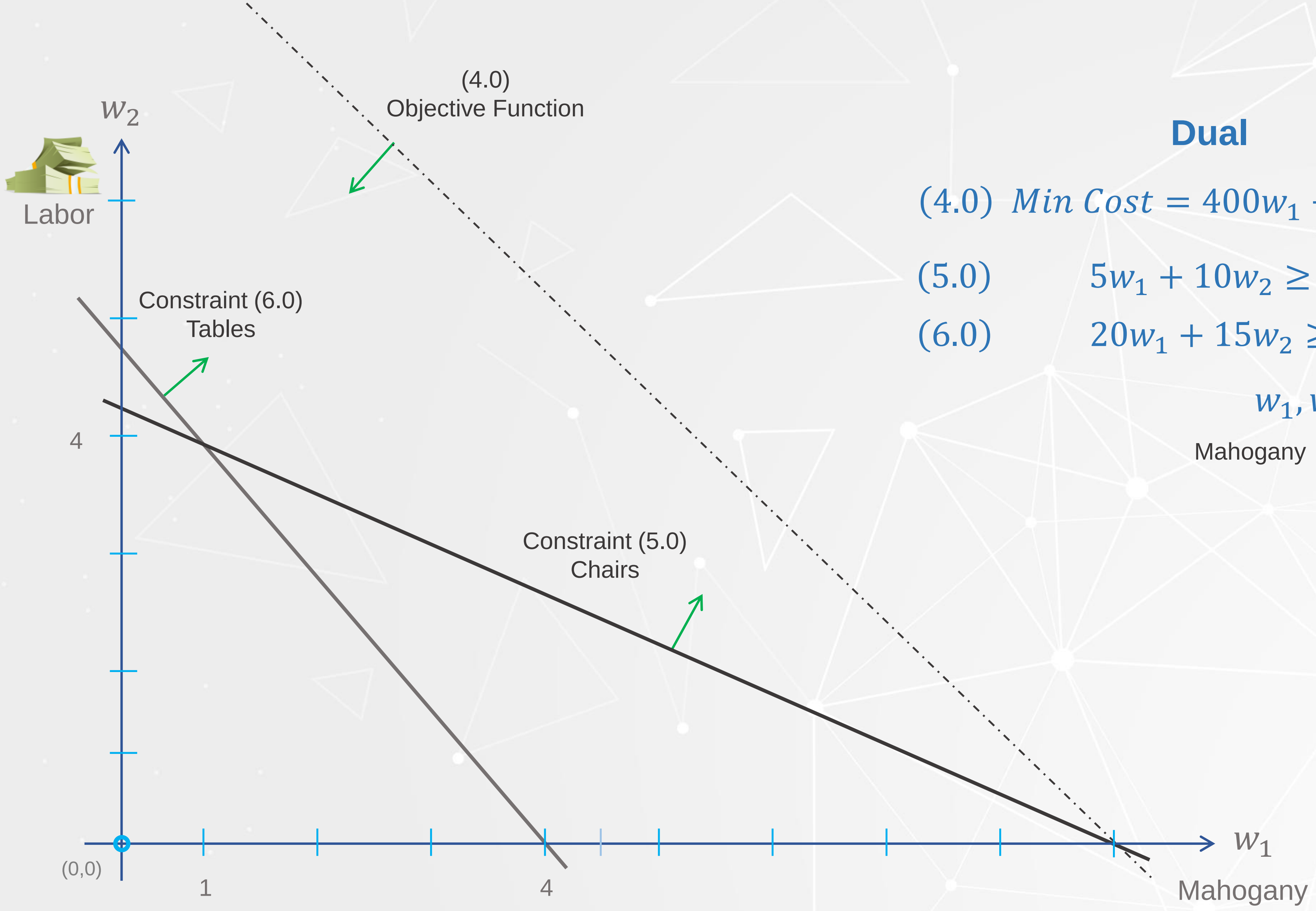
(6.0) $20w_1 + 15w_2 \geq 80$ Tables

$w_1, w_2 \geq 0$

Mahogany Labor

- In this dual problem, the decision variable w1 represents the opportunity cost of the mahogany resource, and w2 is the opportunity cost of the labor resource. These decision variables are the shadow prices of mahogany and labor capacity.
- Notice the switch between the objective function coefficients and the right hand sides of the primal and dual problems.
- Also, notice that the rows of the primal problem are the columns of the dual problem. This means that inequalities (5.0) and (6.0) ensures that the opportunity costs of consumption of resources per unit of production of chairs and tables, respectively, should be at least the value of their price. The objective is to minimize the resource opportunity costs.

Furniture Dual Problem: Graphical solution



Dual

(4.0) $Min\ Cost = 400w_1 + 450w_2$

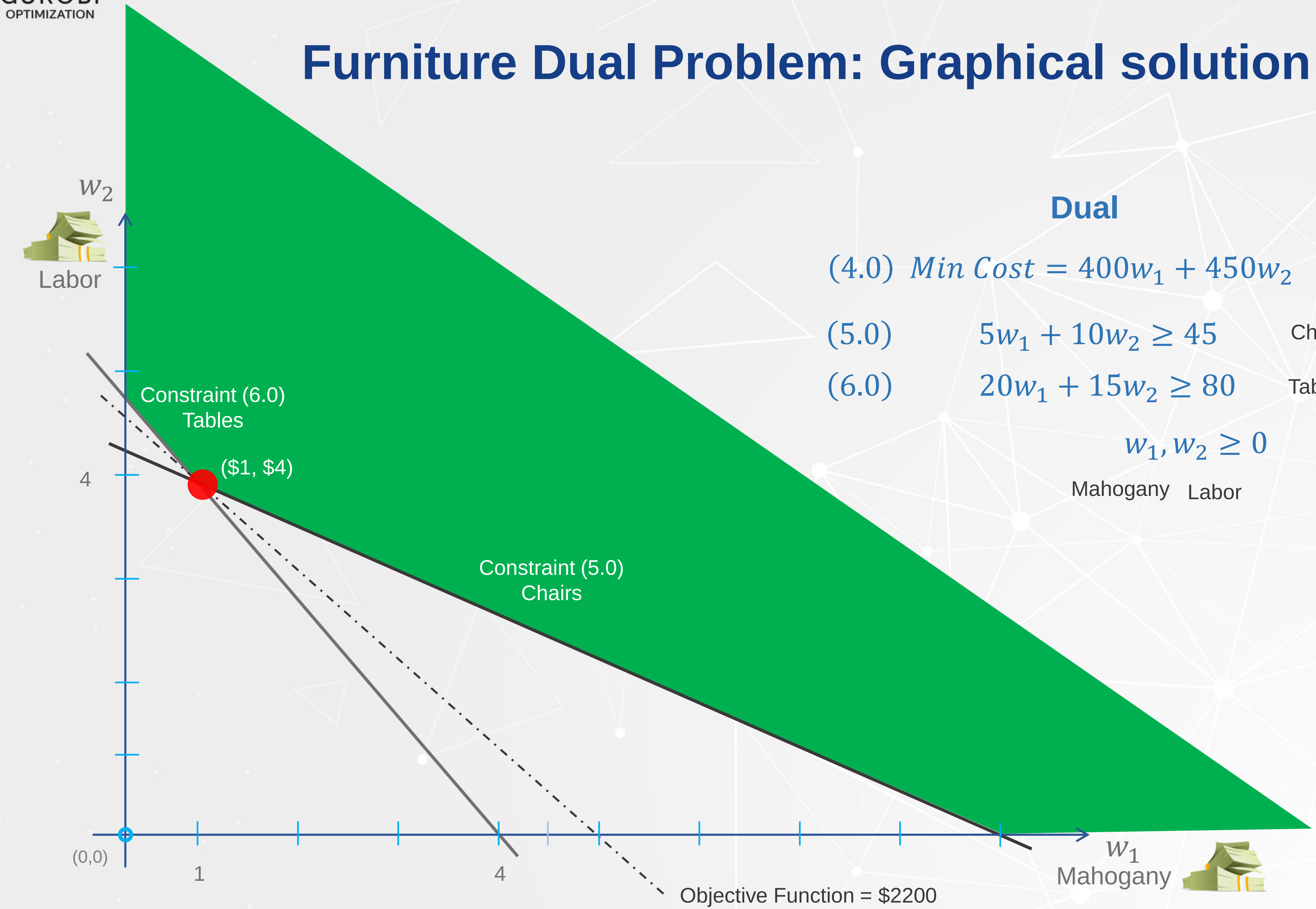
(5.0) $5w_1 + 10w_2 \geq 45$ Chairs

(6.0) $20w_1 + 15w_2 \geq 80$ Tables

$w_1, w_2 \geq 0$

Mahogany Labor

Furniture Dual Problem: Graphical solution



Dual

(4.0) $Min\ Cost = 400w_1 + 450w_2$

(5.0) $5w_1 + 10w_2 \geq 45$ Chairs

(6.0) $20w_1 + 15w_2 \geq 80$ Tables

$w_1, w_2 \geq 0$

Mahogany Labor

Furniture Dual Problem: Parametrized

Furniture dual problem

Let $price_p$ be the price of product $p \in products = \{chairs, \ tables\}$, and let $capacity_r$ be the capacity available of resource $r \in resources = \{mahogany, \ labor\}$. Let $bom_{r,j}$ be the amount of resource r required by product p.

Let $shadowPrice_r$ be the shadow price, or opportunity cost, of resource $r \in resoruces = \{mahogany, \ labor\}$

$$Min \sum_{r \in resource} capacity_r shadowPrice_r$$

$$Subject \ to:$$

$$\sum_{r \in resource} bom_{r,p} shadowPrice_r \geq price_p \quad \forall \, p \in products$$

$$shadowPrice_r \geq 0 \quad \forall \, r \in resources$$

Furniture Dual Problem: Solved with Gurobi ..1

The data of the Furniture Dual problem **is identical** to the original Furniture problem –called Primal Problem.

```
# resources data
resources, capacity = multidict({
  'mahogany': 400,
  'labor':  450 })
```

```
# products data,
products, price = multidict({
  'chair': 45,
  'table':  80 })
```

```
# Bill of materials: resources required by each product
bom = {
  ('mahogany', 'chair'):  5,
  ('mahogany', 'table'): 20,
  ('labor',    'chair'): 10,
  ('labor',    'table'): 15 }
```

Furniture Dual Problem: Solved with Gurobi ..2

The right hand side of the constraints are the price of each product. The left hand side is the opportunity cost of consuming each resource when making the products. The sense of the inequalities is greater than equal to have an evaluation of the resource at least equal to the price.

```
# Declare and initialize model
f = Model('Furniture')
```

```
# Create decision variables for the resources capacity
shadowPrice = f.addVars(resources, name="price")
```

$$\sum_{r \in resource} bom_{r,p} shadowPrice_r \geq price_p \quad \forall p \in products$$

```
# Create an object of type list to store the constraints for each product

pro = f.addConstrs(((sum(bom[r,p]*shadowPrice[r] for r in resources) >= price[p]) for p in products), name='V')
```

```
f.setObjective(shadowPrice.prod(capacity))
```

GRB.MINIMZE is the default

Coefficients in the objective function are the resource capacities

Furniture Dual Problem: Solved with Gurobi ..3

```
# save model for inspection
f.write('furnitureDual.lp')
```

```
\ Model Furniture
\ LP format - for model browsing. Use MPS format to capture full model detail.
Minimize
  400 price[mahogany] + 450 price[labor]
Subject To
 V[table]: 20 price[mahogany] + 15 price[labor] >= 80
 V[chair]: 5 price[mahogany] + 10 price[labor] >= 45
Bounds
End
```

Furniture Dual Problem: Solved with Gurobi ..4

```
# run optimization engine
f.optimize()
```

```
Optimize a model with 2 rows, 2 columns and 4 nonzeros
Coefficient statistics:
  Matrix range     [5e+00, 2e+01]
  Objective range  [4e+02, 5e+02]
  Bounds range     [0e+00, 0e+00]
  RHS range        [5e+01, 8e+01]
Presolve time: 0.13s
Presolved: 2 rows, 2 columns, 4 nonzeros

Iteration    Objective       Primal Inf.    Dual Inf.      Time
       0    0.0000000e+00   1.562500e+01   0.000000e+00      0s
       2    2.2000000e+03   0.000000e+00   0.000000e+00      0s

Solved in 2 iterations and 0.16 seconds
Optimal objective  2.200000000e+03
```

Gurobi solver finds the optimal solution of the Furniture dual problem

```
# display optimal values of decision variables
for v in f.getVars():
    if (abs(v.x) > 1e-6):
        print(v.varName, v.x)

# display optimal total revenue value
print('total revenue', f.objVal)
```

```
('price[mahogany]', 0.9999999999999996)
('price[labor]', 4.000000000000001)
('total revenue', 2200.0000000000005)
```

$1.00 The optimal value of the shadow price for mahogany is $1.00

$4.00 The optimal value of the shadow price for labor is $4.00

$2,200 The optimal objective function value is $2,200

Furniture Dual Problem: Solved with Gurobi .. 5

```
# display shadow prices of product constraints
for p in pro:
    if (abs(pro[p].Pi) > 1e-6):
        print(pro[p].ConstrName, pro[p].Pi)
```

```
('V[table]', 14.0)
('V[chair]', 24.000000000000004)
```

- The “shadow prices” of the products’ constraints are 14 tables and 24 chairs. These are the optimal (make) values of the Furniture primal problem. Notice that in both problems, primal and dual, the optimal objective function value is $2,200. This is not a coincidence!!!

Duality in Linear Programming

Remarks:

- In general, it can be shown that the dual of a dual problem is the primal problem, and that when either problem has an optimal solution, the other problem also has an optimal solution, and the optimal objective function value of both problems is the same.

- Another important feature of duality in linear programming is that the optimal solution of the dual problem is contained in the information provided by the simplex method while solving and finding an optimal solution to the primal problem.

- Duality in linear programming provides a good characterization of optimality conditions that can be exploited computationally to solve LP problems efficiently.

GUROBI OPTIMIZATION

Relationship between primal and dual problems

	Primal (maximize)	Dual (minimize)
1)	i'th constraint ≤	i'th variable ≥ 0
2)	i'th constraint ≥	i'th variable ≤ 0
3)	i'th constraint =	i'th variable unrestricted
4)	j'th variable ≥ 0	j'th constraint ≥
5)	j'th variable ≤ 0	j'th constraint ≤
6)	j'th variable unrestricted	j'th constraint =

Primal

$$(1.0) \quad Max\ revenue = 45x_1 + 80x_2$$

$$(2.0) \quad 5x_1 + 20x_2 \leq 400$$ Mahogany

$$(3.0) \quad 10x_1 + 15x_2 \leq 450$$ Labor

$$x_1, x_2 \geq 0$$

Chairs Tables

Dual

$$(4.0) \quad Min\ Cost = 400w_1 + 450w_2$$

$$(5.0) \quad 5w_1 + 10w_2 \geq 45$$ Chairs

$$(6.0) \quad 20w_1 + 15w_2 \geq 80$$ Tables

$$w_1, w_2 \geq 0$$

Mahogany Labor

Remark:
Notice that the relationship between the Furniture primal and dual problems is captured by rows 1) and 4)